

A New Method for the Direct Solution of Second, Third and Fourth Order Differential Equations with Oscillatory Behavior

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ABSTRACT

This paper presents a new block hybrid method for the direct numerical solution of second, third, and fourth-order ordinary differential equations (ODEs) with oscillatory characteristics. Unlike traditional approaches that rely on transforming higher-order equations into systems of first-order equations, the proposed method tackles the equations in their original forms, thereby reducing computational complexity and preserving the physical structure of the models. The method is formulated using power series approximations, collocation, and interpolation techniques to derive continuous implicit schemes and their discrete counterparts. A rigorous analysis of the method's basic properties such as: order, consistency, zero-stability, convergence, and absolute stability is conducted to establish its reliability. To validate the method, several numerical examples are presented. The results demonstrate that the new method offers high accuracy and stability compared to existing techniques, making it a robust and efficient tool for solving higher-order oscillatory differential equations directly.

Keywords: Higher-order differential equations; Block hybrid method; Continuous block method; Oscillatory systems; Numerical solution.

1.0 INTRODUCTION

The resulting higher-order differential equations ultimately give rise to more advanced forms of differential equations, typically expressed in the general format of higher-order equations

$$u^n = f(v, u, u', u'', \dots, u^{(n-1)}), u^{(m-1)}(v_0) = \mu_{m-1}, n = 1, 2, \dots, n \quad (1.1)$$

These are examined in this study due to their considerable importance in the development of mathematical models, particularly through the use of differential equations, as noted by previous researchers.

Two main approaches are commonly used to simulate equation (1.1). The first involves transforming equation (1.1) into an equivalent system of first-order differential equations, which can then be solved using standard techniques for first-order ODEs. The second approach employs a direct method, as suggested by several studies. However, converting an oscillatory differential equation like (1.1) into a first-order system can introduce certain challenges, such as increased computational complexity and time consumption, which may hinder the overall efficiency of the method.

In recent times, significant efforts have been directed toward developing numerical schemes capable of solving equation (1.1) directly using various approaches. Among these, researchers such as those in Sabo et al (2021) have proposed methods specifically designed to address the dynamics of a moving object, which can be formulated as a second-order oscillatory problem of the form

$$u'' = f(v, u, u'), u(v_0) = \mu_0, u'(v_1) = \mu_1 \quad (1.2)$$

Similarly, alternative approaches have been proposed by researchers in [8-10] for solving

$$u''' = f(v, u, u', u''), u(v_0) = \mu_0, u'(v_1) = \mu_1, u''(v_2) = \mu_2 \quad (1.3)$$

Lastly, several researchers Abolarin et al (2022) have directly applied their methods to

$$u'''' = f(v, u, u', u'', u'''), u(v_0) = \mu_0, u'(v_1) = \mu_1, u''(v_2) = \mu_2, u'''(v_3) = \mu_3 \quad (1.4)$$

From this point onward, the newly developed method aims to address the limitations by directly solving equations (1.2), (1.3), and (1.4). In the existing literature, several studies Skwame et al (2024) have proposed methods for the direct solution of these equations, demonstrating computational reliability and improved accuracy.

2.0 METHODOLOGICAL FORMULATION OF THE METHOD

Consider the approximate solution of power series in the

$$y(\tau) = \sum_{j=0}^{\eta+\nu} \sigma_j \tau^j \quad (3.1)$$

where σ_j 's are parameters to be determined, $\sigma \in [a, b]$, η and ν are the respective number of distinct collocation and interpolation points.

Let the solution of equation (3.1) be sought on the partition

$\pi_N : \sigma = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_n < \tau_{n+1} < \tau_N = b$ on the

$$y(\tau) = \sum_{j=0}^{10} \sigma_j \tau^j \quad (3.2)$$

Differentiating (3.1) four times, yield

$$y''''(\tau) = \sum_{j=0}^{10} j(j-1)(j-2)(j-3)\sigma_j \tau^{j-4} \quad (3.3)$$

Substituting (3.3) into (1.3) give

$$\sum_{j=0}^{10} j(j-1)(j-2)(j-3)\sigma_j \tau^{j-4} = f(\tau, y, y', y'', y''') \quad (3.4)$$

Now interpolating equation (3.2) at $\sigma_{n+\eta}, \eta = 0, \frac{1}{3}, \frac{2}{3}, 1$ and collocating (3.4) at $\sigma_{n+\nu}, \nu = 0\left(\frac{1}{3}\right)2$ to give a

system of nonlinear equation in a matrix form as

$$\Omega X = Z \quad (3.5)$$

where

$$\Omega = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{9} & \frac{1}{27} & \frac{1}{81} & \frac{1}{243} & \frac{1}{729} & \frac{1}{2187} & \frac{1}{6561} & \frac{1}{19683} & \frac{1}{59049} \\ 1 & \frac{2}{3} & \frac{4}{9} & \frac{8}{27} & \frac{16}{81} & \frac{32}{243} & \frac{64}{729} & \frac{128}{2187} & \frac{652}{6561} & \frac{512}{19683} & \frac{1024}{59049} \\ 1 & \frac{3}{3} & \frac{9}{9} & \frac{27}{27} & \frac{81}{81} & \frac{243}{243} & \frac{729}{729} & \frac{2187}{2187} & \frac{6561}{6561} & \frac{19683}{19683} & \frac{59049}{59049} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{40}{h^4} & \frac{40}{h^4} & \frac{280}{9h^4} & \frac{560}{27h^4} & \frac{112}{9h^4} & \frac{560}{81h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{160}{h^4} & \frac{160}{h^4} & \frac{2240}{9h^4} & \frac{8960}{27h^4} & \frac{3584}{9h^4} & \frac{35840}{81h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120}{h^4} & \frac{360}{h^4} & \frac{9h^4}{840} & \frac{27h^4}{1680} & \frac{9h^4}{3024} & \frac{81h^4}{5040} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120}{h^4} & \frac{360}{h^4} & \frac{840}{h^4} & \frac{1680}{h^4} & \frac{3024}{h^4} & \frac{5040}{4h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{160}{h^4} & \frac{640}{h^4} & \frac{17920}{9h^4} & \frac{143360}{27h^4} & \frac{114688}{9h^4} & \frac{2293760}{81h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{100}{h^4} & \frac{1000}{h^4} & \frac{35000}{9h^4} & \frac{350000}{27h^4} & \frac{350000}{9h^4} & \frac{8750000}{81h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{240}{h^4} & \frac{1440}{h^4} & \frac{6720}{9h^4} & \frac{26880}{27h^4} & \frac{96768}{18h^4} & \frac{322560}{81h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{240}{h^4} & \frac{1440}{h^4} & \frac{6720}{h^4} & \frac{26880}{h^4} & \frac{96768}{h^4} & \frac{322560}{h^4} \end{bmatrix}$$

$$X = [\sigma_0 \quad \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5 \quad \sigma_6 \quad \sigma_7 \quad \sigma_8 \quad \sigma_9 \quad \sigma_{10}]^T$$

$$Z = \left[y_n \quad y_{n+\frac{1}{3}} \quad y_{n+\frac{2}{3}} \quad y_{n+1} \quad f_n \quad f_{n+\frac{1}{3}} \quad f_{n+\frac{2}{3}} \quad f_{n+1} \quad f_{n+\frac{4}{3}} \quad f_{n+\frac{5}{3}} \quad f_{n+2} \right]^T$$

The unknown values of σ_j 's, $j = 0(1)10$ in (3.5) can be obtained by using Gaussian elimination method and these

values are substituted back into equation (3.1) to produce a continuous implicit scheme with derivatives of the form:

$$y(\xi) = \alpha_0(\xi)y_n + \alpha_{\frac{1}{3}}(\xi)y_{n+\frac{1}{3}} + \alpha_{\frac{2}{3}}(\xi)y_{n+\frac{2}{3}} + \alpha_1(\xi)y_{n+1} + \quad (3.6)$$

$$h^4 \left[\beta_0(\xi)f_n + \beta_{\frac{1}{3}}(\xi)f_{n+\frac{1}{3}} + \beta_{\frac{2}{3}}(\xi)f_{n+\frac{2}{3}} + \beta_1(\xi)f_{n+1} + \beta_{\frac{4}{3}}(\xi)f_{n+\frac{4}{3}} + \beta_{\frac{5}{3}}(\xi)f_{n+\frac{5}{3}} + \beta_2(\xi)f_{n+2} \right]$$

The coefficient of $\alpha_0, \alpha_{\frac{1}{3}}, \alpha_{\frac{2}{3}}, \alpha_1, y_n, y_{n+\frac{1}{3}}, y_{n+\frac{2}{3}}, y_{n+1}, y_{n+\frac{4}{3}}, y_{n+\frac{5}{3}}, y_{n+2}$ are obtained in terms of ξ as

$$\begin{aligned}
\alpha_0 &= 1 - \frac{11}{2}\tau + 9\tau^2 - \frac{9}{2}\tau^3 \\
\alpha_{\frac{1}{5}} &= 9\tau - \frac{45}{2}\tau^2 + \frac{27}{2}\tau^3 \\
\alpha_{\frac{2}{5}} &= -\frac{9}{2}\tau + 18\tau^2 - \frac{27}{2}\tau^3 \\
\alpha_{\frac{3}{5}} &= \tau - \frac{9}{2}\tau^2 + \frac{9}{2}\tau^3 \\
\beta_0 &= \frac{31781}{340200}\tau + \frac{6721}{1814400}\tau^2 - \frac{36887}{2177280}\tau^3 + \frac{1}{24}\tau^4 - \frac{49}{800}\tau^5 + \frac{203}{3600}\tau^6 - \frac{21}{640}\tau^7 - \frac{3}{1280}\tau^8 + \frac{9}{44800}\tau^{10} \\
\beta_{\frac{1}{3}} &= -\frac{15271}{30240}\tau + \frac{111191}{2721600}\tau^2 - \frac{19553}{272160}\tau^3 + \frac{3}{20}\tau^5 - \frac{87}{400}\tau^6 + \frac{87}{560}\tau^7 + \frac{3}{224}\tau^9 - \frac{27}{22400}\tau^{10} \\
\beta_{\frac{2}{3}} &= \frac{119629}{108864}\tau + \frac{83}{34020}\tau^2 + \frac{1327}{62208}\tau^3 - \frac{3}{16}\tau^5 + \frac{117}{320}\tau^6 - \frac{1383}{4480}\tau^7 - \frac{57}{1792}\tau^8 + \frac{27}{8960}\tau^{10} \\
\beta_1 &= -\frac{2941}{2268}\tau + \frac{31}{4536}\tau^2 - \frac{2419}{90720}\tau^3 + \frac{1}{6}\tau^5 - \frac{127}{360}\tau^6 + \frac{93}{280}\tau^7 + \frac{9}{224}\tau^9 - \frac{9}{8960}\tau^{10} \\
\beta_{\frac{4}{3}} &= \frac{15605}{18144}\tau - \frac{4397}{1088640}\tau^2 - \frac{32323}{2177280}\tau^3 - \frac{3}{32}\tau^5 + \frac{33}{160}\tau^6 - \frac{921}{4480}\tau^7 - \frac{51}{1792}\tau^9 + \frac{27}{8960}\tau^{10} \\
\beta_{\frac{5}{3}} &= -\frac{415649}{1360800}\tau + \frac{3581}{2721600}\tau^2 - \frac{641}{136080}\tau^3 + \frac{3}{100}\tau^5 - \frac{27}{400}\tau^6 + \frac{39}{560}\tau^7 - \frac{3}{280}\tau^9 - \frac{27}{22400}\tau^{10} \\
\beta_1 &= \frac{24793}{544320}\tau - \frac{167}{907200}\tau^2 + \frac{283}{435456}\tau^3 - \frac{1}{240}\tau^5 + \frac{137}{14400}\tau^6 - \frac{9}{896}\tau^7 - \frac{3}{1792}\tau^9 + \frac{9}{44800}\tau^{10}
\end{aligned}$$

Evaluating (3.6) at non interpolating point to obtain the continuous form as

$$\left. \begin{aligned}
y_{\frac{n+4}{3}} &= -y_n + 4y_{\frac{n+1}{6}} - 6y_{\frac{n+1}{3}} + 4y_{\frac{n+1}{2}} + h^4 \left(-\frac{1}{76545}f_n + \frac{143}{68040}f_{\frac{n+1}{3}} + \frac{3343}{408240}f_{\frac{n+2}{3}} + \frac{313}{153090}f_{n+1} + \frac{1}{22680}f_{\frac{n+4}{3}} - \frac{1}{40824}f_{\frac{n+5}{3}} + \frac{1}{244944}f_{n+2} \right) \\
y_{\frac{n+5}{3}} &= -4y_n + 15y_{\frac{n+1}{6}} + 20y_{\frac{n+1}{3}} + 10y_{\frac{n+1}{2}} + h^4 \left(-\frac{59}{1224720}f_n + \frac{683}{81648}f_{\frac{n+1}{3}} + \frac{317}{9072}f_{\frac{n+2}{3}} + \frac{1987}{122472}f_{n+1} + \frac{193}{81648}f_{\frac{n+4}{3}} - \frac{19}{136080}f_{\frac{n+5}{3}} + \frac{5}{244944}f_{n+2} \right) \\
y_{n+2} &= -10y_n + 36y_{\frac{n+1}{6}} - 45y_{\frac{n+1}{3}} + 20y_{\frac{n+1}{2}} + h^4 \left(-\frac{1}{9072}f_n + \frac{1417}{68040}f_{\frac{n+1}{3}} + \frac{617}{6804}f_{\frac{n+2}{3}} + \frac{31}{567}f_{n+1} + \frac{473}{27216}f_{\frac{n+4}{3}} + \frac{23}{13608}f_{\frac{n+5}{3}} + \frac{1}{22680}f_{n+2} \right)
\end{aligned} \right\} \quad (3.7)$$

The first, second and third derivative of (3.6) is given by

$$\begin{aligned}
y'(\xi) &= \alpha'_0(\xi)y_n + \alpha'_{\frac{1}{3}}(\xi)y_{\frac{n+1}{3}} + \alpha'_{\frac{2}{3}}(\xi)y_{\frac{n+2}{3}} + \alpha'_1(\xi)y_{n+1} + \\
& h^4 \left[\beta'_0(\xi)f_n + \beta'_{\frac{1}{3}}(\xi)f_{\frac{n+1}{3}} + \beta'_{\frac{2}{3}}(\xi)f_{\frac{n+2}{3}} + \beta'_1(\xi)f_{n+1} + \beta'_{\frac{4}{3}}(\xi)f_{\frac{n+4}{3}} + \beta'_{\frac{5}{3}}(\xi)f_{\frac{n+5}{3}} + \beta'_2(\xi)f_{n+2} \right] \quad (3.8)
\end{aligned}$$

$$\begin{aligned}
y''(\xi) &= \alpha''_0(\xi)y_n + \alpha''_{\frac{1}{3}}(\xi)y_{\frac{n+1}{3}} + \alpha''_{\frac{2}{3}}(\xi)y_{\frac{n+2}{3}} + \alpha''_1(\xi)y_{n+1} + \\
& h^4 \left[\beta''_0(\xi)f_n + \beta''_{\frac{1}{3}}(\xi)f_{\frac{n+1}{3}} + \beta''_{\frac{2}{3}}(\xi)f_{\frac{n+2}{3}} + \beta''_1(\xi)f_{n+1} + \beta''_{\frac{4}{3}}(\xi)f_{\frac{n+4}{3}} + \beta''_{\frac{5}{3}}(\xi)f_{\frac{n+5}{3}} + \beta''_2(\xi)f_{n+2} \right] \quad (3.9)
\end{aligned}$$

$$\begin{aligned}
y'''(\xi) &= \alpha'''_0(\xi)y_n + \alpha'''_{\frac{1}{3}}(\xi)y_{\frac{n+1}{3}} + \alpha'''_{\frac{2}{3}}(\xi)y_{\frac{n+2}{3}} + \alpha'''_1(\xi)y_{n+1} + \\
& h^4 \left[\beta'''_0(\xi)f_n + \beta'''_{\frac{1}{3}}(\xi)f_{\frac{n+1}{3}} + \beta'''_{\frac{2}{3}}(\xi)f_{\frac{n+2}{3}} + \beta'''_1(\xi)f_{n+1} + \beta'''_{\frac{4}{3}}(\xi)f_{\frac{n+4}{3}} + \beta'''_{\frac{5}{3}}(\xi)f_{\frac{n+5}{3}} + \beta'''_2(\xi)f_{n+2} \right] \quad (3.10)
\end{aligned}$$

Evaluating equations (3.8) to (3.10) of continuous scheme at all the points $0\left(\frac{1}{3}\right)2$ to gives discrete scheme and its

derivatives can be written explicit in the block form as

$$\begin{aligned}
y'_n &= -\frac{11}{2}y_n + 9y_{n+\frac{1}{3}} - \frac{9}{2}y_{n+\frac{2}{3}} + y_{n+1} + h^3 \left(-\frac{451}{1360800}f_n - \frac{41}{6048}f_{n+\frac{1}{3}} - \frac{1093}{544320}f_{n+\frac{2}{3}} - \frac{1}{3240}f_{n+1} + \frac{11}{45360}f_{n+\frac{4}{3}} - \frac{17}{194400}f_{n+\frac{5}{3}} + \frac{1}{77760}f_{n+2} \right) \\
y'_{n+\frac{1}{3}} &= -y_n - \frac{3}{2}y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}} - \frac{1}{2}y_{n+1} + h^3 \left(-\frac{1}{4898880}f_n + \frac{4573}{2721600}f_{n+\frac{1}{3}} + \frac{2743}{1632960}f_{n+\frac{2}{3}} - \frac{1013}{2449440}f_{n+1} + \frac{101}{544320}f_{n+\frac{4}{3}} - \frac{17}{326592}f_{n+\frac{5}{3}} + \frac{163}{24494400}f_{n+2} \right) \\
y'_{n+\frac{2}{3}} &= \frac{1}{2}y_n - 3y_{n+\frac{1}{3}} + \frac{3}{2}y_{n+\frac{2}{3}} + y_{n+1} + h^3 \left(\frac{323}{24494400}f_n - \frac{4481}{4082400}f_{n+\frac{1}{3}} - \frac{619}{272160}f_{n+\frac{2}{3}} + \frac{13}{30618}f_{n+1} - \frac{331}{1632960}f_{n+\frac{4}{3}} + \frac{79}{1360800}f_{n+\frac{5}{3}} - \frac{23}{3061800}f_{n+2} \right) \\
y'_{n+1} &= -y_n + \frac{9}{2}y_{n+\frac{1}{3}} - 9y_{n+\frac{2}{3}} + \frac{11}{2}y_{n+1} + h^3 \left(-\frac{1}{48600}f_n + \frac{5899}{2721600}f_{n+\frac{1}{3}} + \frac{629}{90720}f_{n+\frac{2}{3}} + \frac{1}{30240}f_{n+1} + \frac{1}{4860}f_{n+\frac{4}{3}} - \frac{1}{14400}f_{n+\frac{5}{3}} + \frac{13}{1360800}f_{n+2} \right) \\
y'_{n+\frac{4}{3}} &= \frac{11}{2}y_n + 21y_{n+\frac{1}{3}} - \frac{76}{2}y_{n+\frac{2}{3}} + 13y_{n+1} + h^3 \left(-\frac{109}{1749600}f_n + \frac{15611}{1360800}f_{n+\frac{1}{3}} + \frac{11059}{233280}f_{n+\frac{2}{3}} + \frac{1561}{87480}f_{n+1} + \frac{83}{136080}f_{n+\frac{4}{3}} - \frac{529}{4082400}f_{n+\frac{5}{3}} + \frac{487}{24494400}f_{n+2} \right) \\
y'_{n+\frac{5}{3}} &= -13y_n + \frac{93}{2}y_{n+\frac{1}{3}} - 57y_{n+\frac{2}{3}} + \frac{47}{2}y_{n+1} + h^3 \left(-\frac{3673}{24494400}f_n + \frac{44287}{1632960}f_{n+\frac{1}{3}} + \frac{64417}{544320}f_{n+\frac{2}{3}} + \frac{179519}{2449440}f_{n+1} + \frac{31211}{1632960}f_{n+\frac{4}{3}} - \frac{49}{388800}f_{n+\frac{5}{3}} + \frac{307}{4898880}f_{n+2} \right) \\
y'_{n+2} &= -\frac{47}{2}y_n + 81y_{n+\frac{1}{3}} - \frac{189}{2}y_{n+\frac{2}{3}} + 37y_{n+1} + h^3 \left(-\frac{19}{77760}f_n + \frac{9487}{194440}f_{n+\frac{1}{3}} + \frac{60113}{272160}f_{n+\frac{2}{3}} + \frac{133}{810}f_{n+1} + \frac{41651}{544320}f_{n+\frac{4}{3}} + \frac{4687}{272160}f_{n+\frac{5}{3}} + \frac{139}{340200}f_{n+2} \right)
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
y''_n &= 18y_n - 45y_{n+\frac{1}{3}} + 36y_{n+\frac{2}{3}} - 9y_{n+1} + h^2 \left(\frac{6721}{90720}f_n + \frac{111191}{1360800}f_{n+\frac{1}{3}} + \frac{83}{17010}f_{n+\frac{2}{3}} + \frac{31}{2268}f_{n+1} - \frac{4397}{5443200}f_{n+\frac{4}{3}} + \frac{3581}{1360800}f_{n+\frac{5}{3}} - \frac{167}{453600}f_{n+2} \right) \\
y''_{n+\frac{1}{3}} &= 9y_n - 18y_{n+\frac{1}{3}} + 9y_{n+\frac{2}{3}} + h^2 \left(-\frac{341}{1360800}f_n - \frac{8101}{907200}f_{n+\frac{1}{3}} + \frac{19}{68040}f_{n+\frac{2}{3}} - \frac{173}{272160}f_{n+1} + \frac{1}{2592}f_{n+\frac{4}{3}} - \frac{49}{388800}f_{n+\frac{5}{3}} + \frac{1}{56700}f_{n+2} \right) \\
y''_{n+\frac{2}{3}} &= -9y_n + 36y_{n+\frac{1}{3}} - 45y_{n+\frac{2}{3}} + 18y_{n+1} + h^2 \left(\frac{1}{90720}f_n - \frac{509}{5443200}f_{n+\frac{1}{3}} - \frac{1553}{725760}f_{n+\frac{2}{3}} - \frac{23}{272160}f_{n+1} - \frac{1}{217728}f_{n+\frac{4}{3}} + \frac{1}{259200}f_{n+\frac{5}{3}} - \frac{1}{1555200}f_{n+2} \right) \\
y''_{n+1} &= -9y_n + 36y_{n+\frac{1}{3}} - 45y_{n+\frac{2}{3}} + 18y_{n+1} + h^2 \left(-\frac{109}{90720}f_n + \frac{51577}{2721600}f_{n+\frac{1}{3}} + \frac{39883}{544320}f_{n+\frac{2}{3}} + \frac{901}{90720}f_{n+1} - \frac{17}{544320}f_{n+\frac{4}{3}} - \frac{503}{2721600}f_{n+\frac{5}{3}} + \frac{31}{907200}f_{n+2} \right) \\
y''_{n+\frac{4}{3}} &= -18y_n + 63y_{n+\frac{1}{3}} - 72y_{n+\frac{2}{3}} + 27y_{n+1} + h^2 \left(-\frac{547}{2721600}f_n + \frac{16997}{453600}f_{n+\frac{1}{3}} + \frac{45469}{272160}f_{n+\frac{2}{3}} + \frac{926}{8505}f_{n+1} + \frac{103}{8640}f_{n+\frac{4}{3}} - \frac{1619}{1360800}f_{n+\frac{5}{3}} + \frac{29}{226800}f_{n+2} \right) \\
y''_{n+\frac{5}{3}} &= -27y_n + 90y_{n+\frac{1}{3}} - 99y_{n+\frac{2}{3}} + 36y_{n+1} + h^2 \left(-\frac{17}{75600}f_n + \frac{151457}{2721600}f_{n+\frac{1}{3}} + \frac{23701}{90720}f_{n+\frac{2}{3}} + \frac{11855}{54432}f_{n+1} + \frac{15587}{136080}f_{n+\frac{4}{3}} + \frac{2593}{302400}f_{n+\frac{5}{3}} - \frac{251}{1360800}f_{n+2} \right) \\
y''_{n+2} &= -36y_n + 117y_{n+\frac{1}{3}} - 126y_{n+\frac{2}{3}} + 45y_{n+1} + h^2 \left(-\frac{11}{16800}f_n + \frac{104411}{1360800}f_{n+\frac{1}{3}} + \frac{188647}{544320}f_{n+\frac{2}{3}} + \frac{7741}{22680}f_{n+1} + \frac{5795}{27216}f_{n+\frac{4}{3}} + \frac{159941}{1360800}f_{n+\frac{5}{3}} + \frac{107}{14400}f_{n+2} \right)
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
y'''_n &= -27y_n + 81y_{n+\frac{1}{3}} - 81y_{n+\frac{2}{3}} + 27y_{n+1} + h \left(-\frac{36887}{362880}f_n - \frac{19553}{45360}f_{n+\frac{1}{3}} + \frac{1327}{10368}f_{n+\frac{2}{3}} - \frac{2419}{15120}f_{n+1} + \frac{32324}{362880}f_{n+\frac{4}{3}} - \frac{641}{22680}f_{n+\frac{5}{3}} + \frac{283}{72576}f_{n+2} \right) \\
y'''_{n+\frac{1}{3}} &= -27y_n + 81y_{n+\frac{1}{3}} - 81y_{n+\frac{2}{3}} + 27y_{n+1} + h \left(-\frac{143}{40320}f_n - \frac{655}{9072}f_{n+\frac{1}{3}} - \frac{46477}{362880}f_{n+\frac{2}{3}} + \frac{2119}{45360}f_{n+1} - \frac{1157}{51840}f_{n+\frac{4}{3}} + \frac{37}{5670}f_{n+\frac{5}{3}} - \frac{311}{362880}f_{n+2} \right) \\
y'''_{n+\frac{2}{3}} &= -27y_n + 81y_{n+\frac{1}{3}} - 81y_{n+\frac{2}{3}} + 27y_{n+1} + h \left(-\frac{432}{362880}f_n + \frac{3007}{45360}f_{n+\frac{1}{3}} + \frac{47501}{362880}f_{n+\frac{2}{3}} - \frac{389}{9072}f_{n+1} + \frac{6499}{362880}f_{n+\frac{4}{3}} - \frac{113}{22680}f_{n+\frac{5}{3}} + \frac{11}{17280}f_{n+2} \right) \\
y'''_{n+1} &= -27y_n + 81y_{n+\frac{1}{3}} - 81y_{n+\frac{2}{3}} + 27y_{n+1} + h \left(\frac{103}{362880}f_n + \frac{331}{6480}f_{n+\frac{1}{3}} + \frac{109139}{362880}f_{n+\frac{2}{3}} + \frac{2477}{15120}f_{n+1} - \frac{7043}{362880}f_{n+\frac{4}{3}} + \frac{11}{2835}f_{n+\frac{5}{3}} - \frac{151}{362880}f_{n+2} \right) \\
y'''_{n+\frac{4}{3}} &= -27y_n + 81y_{n+\frac{1}{3}} - 81y_{n+\frac{2}{3}} + 27y_{n+1} + h \left(-\frac{31}{40320}f_n + \frac{2719}{45360}f_{n+\frac{1}{3}} + \frac{95597}{362880}f_{n+\frac{2}{3}} - \frac{2401}{6480}f_{n+1} + \frac{10919}{72576}f_{n+\frac{4}{3}} - \frac{257}{22680}f_{n+\frac{5}{3}} + \frac{391}{362880}f_{n+2} \right) \\
y'''_{n+\frac{5}{3}} &= -27y_n + 81y_{n+\frac{1}{3}} - 81y_{n+\frac{2}{3}} + 27y_{n+1} + h \left(\frac{263}{362880}f_n + \frac{2197331}{45360}f_{n+\frac{1}{3}} + \frac{22039}{72576}f_{n+\frac{2}{3}} + \frac{12743}{45360}f_{n+1} - \frac{148573}{362880}f_{n+\frac{4}{3}} + \frac{103}{810}f_{n+\frac{5}{3}} - \frac{89}{24192}f_{n+2} \right) \\
y'''_{n+2} &= -27y_n + 81y_{n+\frac{1}{3}} - 81y_{n+\frac{2}{3}} + 27y_{n+1} + h \left(-\frac{209}{51840}f_n + \frac{755}{9072}f_{n+\frac{1}{3}} + \frac{69773}{362880}f_{n+\frac{2}{3}} + \frac{7373}{15120}f_{n+1} + \frac{55651}{362880}f_{n+\frac{4}{3}} + \frac{11023}{22680}f_{n+\frac{5}{3}} + \frac{36839}{362880}f_{n+2} \right)
\end{aligned} \tag{3.13}$$

Equations (3.7), (3.11) to (3.13) are combined together in matrix form and by using matrix inversion to gives the following new method as

$$y_{n+\frac{1}{3}} = y_n + \frac{1}{3} h^1 y'_n + \frac{1}{18} h^2 y''_n + \frac{1}{162} h^3 y'''_n + \frac{95929}{293932800} h^4 f_n + \frac{4001}{10497600} h^4 f_{n+\frac{1}{3}} -$$

$$\begin{aligned}
& \frac{23033}{58786560} h^4 f_{n+\frac{2}{3}} + \frac{811}{2449440} h^4 f_{n+1} - \frac{10693}{58786560} h^4 f_{n+\frac{4}{3}} + \frac{4219}{73483200} h^4 f_{n+\frac{5}{3}} - \\
& \frac{2323}{293932800} h^4 f_{n+2} \\
y_{n+\frac{2}{3}} &= y_n + \frac{2}{3} h^1 y'_n + \frac{9}{2} h^2 y''_n + \frac{4}{81} h^3 y'''_n + \frac{4127}{1148175} h^4 f_n + \frac{8782}{1148175} h^4 f_{n+\frac{1}{3}} - \frac{199}{32805} h^4 f_{n+\frac{2}{3}} \\
& + \frac{388}{76545} h^4 f_{n+1} - \frac{127}{45927} h^4 f_{n+\frac{4}{3}} + \frac{998}{1148175} h^4 f_{n+\frac{5}{3}} - \frac{137}{1148175} h^4 f_{n+2} \\
y_{n+1} &= y_n + h^1 y'_n + \frac{1}{2} h^2 y''_n + \frac{1}{6} h^3 y'''_n + \frac{5471}{403200} h^4 f_n + \frac{423}{11200} h^4 f_{n+\frac{1}{3}} - \frac{39}{1792} h^4 f_{n+\frac{2}{3}} \\
& + \frac{29}{1440} h^4 f_{n+1} - \frac{99}{8960} h^4 f_{n+\frac{4}{3}} + \frac{39}{11200} h^4 f_{n+\frac{5}{3}} - \frac{193}{403200} h^4 f_{n+2} \\
y_{n+\frac{4}{3}} &= y_n + \frac{4}{3} h^1 y'_n + \frac{8}{9} h^2 y''_n + \frac{32}{81} h^3 y'''_n + \frac{7808}{229635} h^4 f_n + \frac{124928}{1148175} h^4 f_{n+\frac{1}{3}} - \frac{10112}{229635} h^4 f_{n+\frac{2}{3}} \\
& + \frac{4096}{76545} h^4 f_{n+1} - \frac{928}{32805} h^4 f_{n+\frac{4}{3}} + \frac{2048}{229635} h^4 f_{n+\frac{5}{3}} - \frac{1408}{1148175} h^4 f_{n+2} \\
y_{n+\frac{5}{3}} &= y_n + \frac{5}{3} h^1 y'_n + \frac{25}{18} h^2 y''_n + \frac{125}{162} h^3 y'''_n + \frac{807125}{11757312} h^4 f_n + \frac{701875}{2939328} h^4 f_{n+\frac{1}{3}} - \\
& \frac{790625}{11757312} h^4 f_{n+\frac{2}{3}} + \frac{59375}{489888} h^4 f_{n+1} - \frac{653125}{11757312} h^4 f_{n+\frac{4}{3}} + \frac{7625}{419904} h^4 f_{n+\frac{5}{3}} - \frac{29375}{11757312} h^4 f_{n+2} \\
y_{n+2} &= y_n + 2 h^1 y'_n + 2 h^2 y''_n + \frac{4}{3} h^3 y'''_n + \frac{191}{1575} h^4 f_n + \frac{78}{175} h^4 f_{n+\frac{1}{3}} - \frac{3}{35} h^4 f_{n+\frac{2}{3}} \\
& + \frac{76}{175} h^4 f_{n+1} - \frac{3}{35} h^4 f_{n+\frac{4}{3}} + \frac{6}{175} h^4 f_{n+\frac{5}{3}} - \frac{1}{225} h^4 f_{n+2} \\
y'_{n+\frac{1}{3}} &= y'_n + \frac{1}{3} h^1 y''_n + \frac{1}{18} h^2 y'''_n + \frac{343801}{97977600} h^3 f_n + \frac{6031}{1166400} h^3 f_{n+\frac{1}{3}} - \frac{32981}{6531840} h^3 f_{n+\frac{2}{3}} \\
& + \frac{5177}{1224720} h^3 f_{n+1} - \frac{15107}{6531840} h^3 f_{n+\frac{4}{3}} + \frac{5947}{8164800} h^3 f_{n+\frac{5}{3}} - \frac{9809}{97977600} h^3 f_{n+2} \\
y'_{n+\frac{2}{3}} &= y'_n + \frac{2}{3} h^1 y''_n + \frac{9}{2} h^2 y'''_n + \frac{6887}{382725} h^3 f_n + \frac{5996}{127575} h^3 f_{n+\frac{1}{3}} - \frac{233}{7290} h^3 f_{n+\frac{2}{3}} \\
& + \frac{416}{15309} h^3 f_{n+1} - \frac{379}{25515} h^3 f_{n+\frac{4}{3}} + \frac{596}{127575} h^3 f_{n+\frac{5}{3}} - \frac{491}{765450} h^3 f_{n+2} \\
y'_{n+1} &= y'_n + h^1 y''_n + \frac{1}{2} h^2 y'''_n + \frac{1959}{44800} h^3 f_n + \frac{1599}{11200} h^3 f_{n+\frac{1}{3}} - \frac{537}{8960} h^3 f_{n+\frac{2}{3}} + \frac{1}{15} h^3 f_{n+1} \\
& - \frac{327}{8960} h^3 f_{n+\frac{4}{3}} + \frac{129}{11200} h^3 f_{n+\frac{5}{3}} - \frac{71}{44800} h^3 f_{n+2} \\
y'_{n+\frac{4}{3}} &= y'_n + \frac{4}{3} h^1 y''_n + \frac{32}{81} h^2 y'''_n + \frac{30904}{382725} h^3 f_n + \frac{37312}{127575} h^3 f_{n+\frac{1}{3}} - \frac{1808}{25515} h^3 f_{n+\frac{2}{3}} + \\
& \frac{2176}{15309} h^3 f_{n+1} - \frac{248}{3645} h^3 f_{n+\frac{4}{3}} + \frac{2752}{127575} h^3 f_{n+\frac{5}{3}} - \frac{1136}{382725} h^3 f_{n+2}
\end{aligned}$$

$$y'_{n+\frac{5}{3}} = y'_n + \frac{5}{3} h^1 y''_n + \frac{25}{18} h^2 y'''_n + \frac{505625}{3919104} h^3 f_n + \frac{162125}{326592} h^3 f_{n+\frac{1}{3}} - \frac{85625}{1306368} h^3 f_{n+\frac{2}{3}} + \frac{66875}{244944} h^3 f_{n+1} - \frac{119375}{1306368} h^3 f_{n+\frac{4}{3}} + \frac{1625}{46656} h^3 f_{n+\frac{5}{3}} - \frac{18625}{3919104} h^3 f_{n+2}$$

$$y'_{n+2} = y'_n + 2 h^1 y''_n + 2 h^2 y'''_n + \frac{33}{175} h^3 f_n + \frac{132}{175} h^3 f_{n+\frac{1}{3}} - \frac{3}{70} h^3 f_{n+\frac{2}{3}} + \frac{16}{35} h^3 f_{n+1} - \frac{3}{35} h^3 f_{n+\frac{4}{3}} + \frac{12}{175} h^3 f_{n+\frac{5}{3}} - \frac{1}{150} h^3 f_{n+2}$$

$$y''_{n+\frac{1}{3}} = y''_n + \frac{1}{3} h^1 y'''_n + \frac{28549}{1088640} h^2 f_n + \frac{275}{5184} h^2 f_{n+\frac{1}{3}} - \frac{5717}{120960} h^2 f_{n+\frac{2}{3}} + \frac{10621}{272160} h^2 f_{n+1} + \frac{7703}{362880} h^2 f_{n+\frac{4}{3}} + \frac{403}{60480} h^2 f_{n+\frac{5}{3}} - \frac{199}{217728} h^2 f_{n+2}$$

$$y''_{n+\frac{2}{3}} = y''_n + \frac{2}{3} h^1 y'''_n + \frac{1027}{17010} h^2 f_n + \frac{194}{945} h^2 f_{n+\frac{1}{3}} - \frac{8}{81} h^2 f_{n+\frac{2}{3}} + \frac{788}{8505} h^2 f_{n+1} + \frac{97}{1890} h^2 f_{n+\frac{4}{3}} + \frac{46}{2835} h^2 f_{n+\frac{5}{3}} - \frac{19}{8505} h^2 f_{n+2}$$

$$y''_{n+1} = y''_n + h^1 y'''_n + \frac{253}{2688} h^2 f_n + \frac{165}{448} h^2 f_{n+\frac{1}{3}} - \frac{267}{4480} h^2 f_{n+\frac{2}{3}} + \frac{5}{32} h^2 f_{n+1} + \frac{363}{4480} h^2 f_{n+\frac{4}{3}} + \frac{57}{2240} h^2 f_{n+\frac{5}{3}} - \frac{47}{13440} h^2 f_{n+2}$$

$$y''_{n+\frac{4}{3}} = y''_n + \frac{4}{3} h^1 y'''_n + \frac{1088}{8505} h^2 f_n + \frac{1504}{2835} h^2 f_{n+\frac{1}{3}} - \frac{9}{945} h^2 f_{n+\frac{2}{3}} + \frac{2624}{8505} h^2 f_{n+1} - \frac{8}{81} h^2 f_{n+\frac{4}{3}} + \frac{32}{945} h^2 f_{n+\frac{5}{3}} - \frac{8}{1701} h^2 f_{n+2}$$

$$y''_{n+\frac{5}{3}} = y''_n + \frac{5}{3} h^1 y'''_n + \frac{35225}{217728} h^2 f_n + \frac{8375}{12096} h^2 f_{n+\frac{1}{3}} + \frac{3125}{72576} h^2 f_{n+\frac{2}{3}} + \frac{25625}{54432} h^2 f_{n+1} - \frac{625}{24192} h^2 f_{n+\frac{4}{3}} + \frac{275}{5184} h^2 f_{n+\frac{5}{3}} - \frac{1375}{217728} h^2 f_{n+2}$$

$$y''_{n+2} = y''_n + 2 h^1 y'''_n + \frac{41}{210} h^2 f_n + \frac{6}{7} h^2 f_{n+\frac{1}{3}} + \frac{3}{35} h^2 f_{n+\frac{2}{3}} + \frac{68}{105} h^2 f_{n+1} + \frac{3}{70} h^2 f_{n+\frac{4}{3}} + \frac{6}{35} h^2 f_{n+\frac{5}{3}}$$

$$y'''_{n+\frac{1}{3}} = y'''_n + \frac{19087}{181440} h^1 f_n + \frac{2713}{7560} h^1 f_{n+\frac{1}{3}} - \frac{15487}{60480} h^1 f_{n+\frac{2}{3}} + \frac{586}{2835} h^1 f_{n+1} + \frac{6737}{60480} h^1 f_{n+\frac{4}{3}} + \frac{263}{7560} h^1 f_{n+\frac{5}{3}} - \frac{863}{181440} h^1 f_{n+2}$$

$$y'''_{n+\frac{2}{3}} = y'''_n + \frac{1139}{11340} h^1 f_n + \frac{94}{189} h^1 f_{n+\frac{1}{3}} + \frac{11}{3780} h^1 f_{n+\frac{2}{3}} + \frac{332}{2835} h^1 f_{n+1} - \frac{269}{3780} h^1 f_{n+\frac{4}{3}} + \frac{22}{945} h^1 f_{n+\frac{5}{3}} - \frac{37}{11340} h^1 f_{n+2}$$

$$\begin{aligned}
y'''_{n+1} &= y'''_n + \frac{137}{1344} h^1 f_n + \frac{27}{56} h^1 f_{n+\frac{1}{3}} + \frac{387}{2240} h^1 f_{n+\frac{2}{3}} + \frac{34}{105} h^1 f_{n+1} - \frac{243}{2240} h^1 f_{n+\frac{4}{3}} \\
&\quad + \frac{9}{280} h^1 f_{n+\frac{5}{3}} - \frac{29}{6720} h^1 f_{n+2} \\
y'''_{n+\frac{4}{3}} &= y'''_n + \frac{286}{2835} h^1 f_n + \frac{464}{945} h^1 f_{n+\frac{1}{3}} + \frac{128}{945} h^1 f_{n+\frac{2}{3}} + \frac{1504}{2835} h^1 f_{n+1} + \frac{58}{945} h^1 f_{n+\frac{4}{3}} \\
&\quad + \frac{16}{945} h^1 f_{n+\frac{5}{3}} - \frac{8}{2835} h^1 f_{n+2} \\
y'''_{n+\frac{5}{3}} &= y'''_n + \frac{3715}{36288} h^1 f_n + \frac{725}{1512} h^1 f_{n+\frac{1}{3}} + \frac{2125}{12096} h^1 f_{n+\frac{2}{3}} + \frac{250}{567} h^1 f_{n+1} + \\
&\quad \frac{3875}{12096} h^1 f_{n+\frac{4}{3}} + \frac{235}{1512} h^1 f_{n+\frac{5}{3}} - \frac{275}{36288} h^1 f_{n+2} \\
y'''_{n+2} &= y'''_n + \frac{41}{420} h^1 f_n + \frac{18}{35} h^1 f_{n+\frac{1}{3}} + \frac{9}{140} h^1 f_{n+\frac{2}{3}} + \frac{68}{105} h^1 f_{n+1} + \frac{9}{140} h^1 f_{n+\frac{4}{3}} \\
&\quad + \frac{18}{35} h^1 f_{n+\frac{5}{3}} + \frac{41}{420} h^1 f_{n+2}
\end{aligned} \tag{3.14}$$

3.0 ANALYSIS OF BASIC PROPERTIES OF THE NEW METHOD

The basic properties of the new method are numerically analyzed in this section.

3.1 Order and Error Constant

Consider the linear operator associated in L associated with the new method defined as

$$L[y(x); h] = \sum_{j=0}^2 \left\{ \alpha_j y(x_n + jh) - \alpha_{vi} y(x_n + vih) - h^d \beta_j y^d(x_n + jh) - h^d \beta_{vi} y^d(x_n + vih) \right\} \tag{3.1}$$

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a, b]$. We expand $y(x_n + jh)$ and $y^d(x_n + jh)$ using a Taylor series about x_n and collecting like terms in h and y to obtain the expression;

$$\ell\{y(x); h\} = C_0 y(x) + C_1 y'(x) + \dots + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) + C_{p+2} h^{p+2} y^{p+2}(x) + \dots \tag{3.2}$$

We consider the linear operator $L[y(t_n); h]$ in (3.1) with the Proposition 3.1 and 3.2 below to determining the order and error constant of the new method Skwame et al (2024).

Proposition 3.1

The linear operator $L[y(t_n); h]$ associate with the local truncation error of the new method is $C_{07} h^{07} y^{07}(t_n) + O(h^{11})$.

Proof

The linear difference operators associated with the new method are given

$$\left. \begin{aligned}
L[y(t_n);h] &= y\left(t_n + \frac{1}{3}h\right) - \left(\alpha_0(x_n) + \alpha_{\frac{1}{3}}\left(x_n + \frac{1}{3}h\right) + \alpha_{\frac{2}{3}}\left(t_n + \frac{2}{3}h\right) + \alpha_1(t_n + h) + h^4 \sum_{i=0}^{\eta} (\beta_i(t)f_{n+i} + \beta_{\eta}(t)f_{n+\eta}) \right) \\
L[y(t_n);h] &= y\left(t_n + \frac{2}{3}h\right) - \left(\alpha_0(x_n) + \alpha_{\frac{1}{3}}\left(x_n + \frac{1}{3}h\right) + \alpha_{\frac{2}{3}}\left(t_n + \frac{2}{3}h\right) + \alpha_1(t_n + h) + h^4 \sum_{i=0}^{\eta} (\beta_i(t)f_{n+i} + \beta_{\eta}(t)f_{n+\eta}) \right) \\
L[y(t_n);h] &= y(t_n + h) - \left(\alpha_0(x_n) + \alpha_{\frac{1}{3}}\left(x_n + \frac{1}{3}h\right) + \alpha_{\frac{2}{3}}\left(t_n + \frac{2}{3}h\right) + \alpha_1(t_n + h) + h^4 \sum_{i=0}^{\eta} (\beta_i(t)f_{n+i} + \beta_{\eta}(t)f_{n+\eta}) \right) \\
L[y(t_n);h] &= y\left(t_n + \frac{4}{3}h\right) - \left(\alpha_0(x_n) + \alpha_{\frac{1}{3}}\left(x_n + \frac{1}{3}h\right) + \alpha_{\frac{2}{3}}\left(t_n + \frac{2}{3}h\right) + \alpha_1(t_n + h) + h^4 \sum_{i=0}^{\eta} (\beta_i(t)f_{n+i} + \beta_{\eta}(t)f_{n+\eta}) \right) \\
L[y(t_n);h] &= y\left(t_n + \frac{5}{3}h\right) - \left(\alpha_0(x_n) + \alpha_{\frac{1}{3}}\left(x_n + \frac{1}{3}h\right) + \alpha_{\frac{2}{3}}\left(t_n + \frac{2}{3}h\right) + \alpha_1(t_n + h) + h^4 \sum_{i=0}^{\eta} (\beta_i(t)f_{n+i} + \beta_{\eta}(t)f_{n+\eta}) \right) \\
L[y(t_n);h] &= y(t_n + 2h) - \left(\alpha_0(x_n) + \alpha_{\frac{1}{3}}\left(x_n + \frac{1}{3}h\right) + \alpha_{\frac{2}{3}}\left(t_n + \frac{2}{3}h\right) + \alpha_1(t_n + h) + h^4 \sum_{i=0}^{\eta} (\beta_i(t)f_{n+i} + \beta_{\eta}(t)f_{n+\eta}) \right)
\end{aligned} \right\} \quad (3.3)$$

Proposition 3.2

The local truncation error of the new method is assume $y(t)$ to be sufficiently differentiable and expanding $y(t_n + qh)$ and $y(t_n + jh)$ about t_n using Taylor series to have

$$\begin{aligned}
L_{\frac{1}{3}}[y(t_n);h] &= (2.9677 \times 10^{-09}), & L_{\frac{2}{3}}[y(t_n);h] &= (4.4229 \times 10^{-08}), & L_1[y(t_n);h] &= (1.7815 \times 10^{-07}), \\
L_{\frac{4}{3}}[y(t_n);h] &= (4.5723 \times 10^{-07}), & L_{\frac{5}{3}}[y(t_n);h] &= (9.3396 \times 10^{-07}), & L_2[y(t_n);h] &= (1.6627 \times 10^{-06})
\end{aligned}$$

Proof

Expand equation (3.3) using proposition 3.2 and then collect the like terms to the power of h gives

$$\begin{aligned}
L_{\frac{1}{3}}[y(t_n);h] &= (2.9677 \times 10^{-09})C_{07}h^{07}y^{07}(t_n) + O(h^{11}) \\
L_{\frac{2}{3}}[y(t_n);h] &= (4.4229 \times 10^{-08})C_{07}h^{07}y^{07}(t_n) + O(h^{11}) \\
L_1[y(t_n);h] &= (1.7815 \times 10^{-07})C_{07}h^{07}y^{07}(t_n) + O(h^{11}) \\
L_{\frac{4}{3}}[y(t_n);h] &= (4.5723 \times 10^{-07})C_{07}h^{07}y^{07}(t_n) + O(h^{11}) \\
L_{\frac{5}{3}}[y(t_n);h] &= (9.3396 \times 10^{-07})C_{07}h^{07}y^{07}(t_n) + O(h^{11}) \\
L_2[y(t_n);h] &= (1.6627 \times 10^{-06})C_{07}h^{07}y^{07}(t_n) + O(h^{11})
\end{aligned}$$

3.2 Consistency

According to [7], The new method is consistent since it of uniform order seven.

3.6.1.3 Zero Stability of equation (3.64)

To determine the zero-stability of the new method, we employ the roots $z_s, s = 1, 2, \dots, n$ of the first characteristic polynomial $\bar{\rho}(z)$, defined by

$$\bar{\rho}(z) = \det [zA^{(0)} - E] \quad (3.4)$$

satisfies $|z_s| \leq 1$ and every root with $|z_s| = 1$ has multiplicity not exceeding the order of the differential equation as $h \rightarrow 0$. Moreover, as $h \rightarrow 0$, $\rho(z) = z^{r-\mu}(z-1)^\mu$, where μ is the order of the differential equation, r is the order of the matrices $A^{(0)}$ and E , given by

$$\rho(z) = z \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} z & 0 & 0 & 0 & 0 & -1 \\ 0 & z & 0 & 0 & 0 & -1 \\ 0 & 0 & z & 0 & 0 & -1 \\ 0 & 0 & 0 & z & 0 & -1 \\ 0 & 0 & 0 & 0 & z & -1 \\ 0 & 0 & 0 & 0 & 0 & z-1 \end{bmatrix} = z^5(z-1)$$

Solving for z gives $z = 0, 0, 0, 0, 0, 1$. Hence, the new method is zero-stable [8].

3.4 Convergence

The new method is convergent, since it satisfied consistent and zero stable by the following theorem Ayinde (2023).

Theorem 3.1:

The necessary and sufficient conditions for a new method to be convergent are that it be consistent and zero-stable.

3.5 Region of Absolute Stability (RAS)

According to Ayinde et al (2023), the Boundary Locus Method were used on the new method to obtained the stability polynomial as

$$\begin{aligned} \bar{h}(w) = & \left(-\frac{182263}{964467000}w^5 + \frac{12163}{96446700}w^6 \right)h^6 + \left(-\frac{178483}{64297800}w^5 - \frac{63701}{26790750}w^6 \right)h^5 + \left(-\frac{4297219}{183708000}w^5 + \frac{85037}{3280500}w^6 \right)h^4 \\ & + \left(-\frac{2785037}{21432600}w^5 - \frac{748471}{5358150}w^6 \right)h^3 + \left(-\frac{6689107}{14288400}w^5 + \frac{315751}{893025}w^6 \right)h^2 + (-w^5 - w^6)h - w^5 + w^6 \end{aligned} \quad (3.5)$$

Using the stability polynomial (3.5), the region of absolute stability of the new method is obtained as

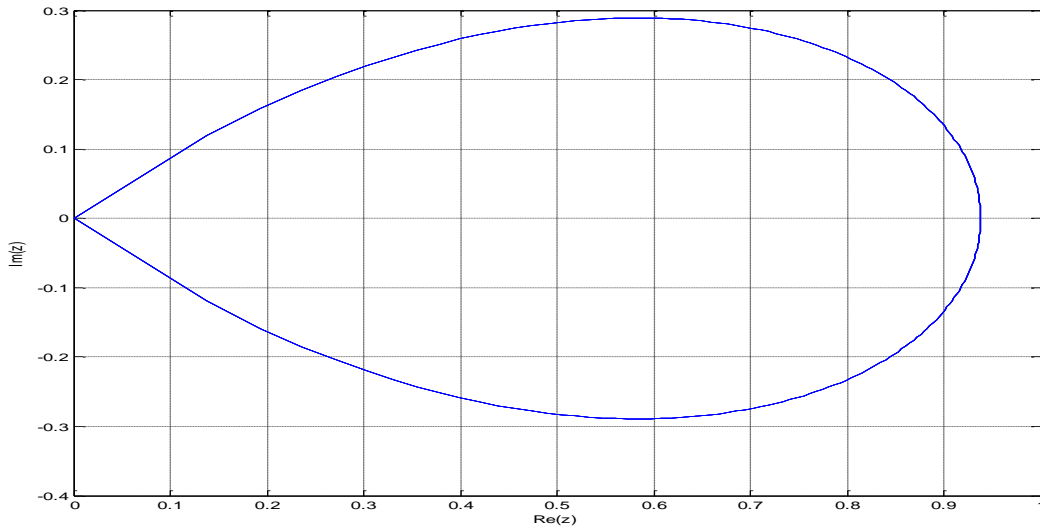


Figure 3.1: Showing an A-stable region of absolute stability

4.0 EXPERIMENTAL EXAMPLES

To demonstrate the effectiveness and accuracy of the proposed multi-order schemes, a series of experimental examples are presented in this section. These examples include second, third, and fourth-order differential equations derived from both physical systems and mathematical models involving linear and nonlinear dynamics. The equations, as outlined in (1.2), (1.3), and (1.4), are solved directly using the new method without transformation to first-order systems. The results are compared to existing methods to evaluate performance in terms of computational efficiency and solution accuracy.

Example 4.1: (Dynamic Problem)

The mass of an object is consider in a dynamic motion that is coined into linear oscillatory form of differential equation (1.2).

A mass of $10kg$ is attached to a spring having a constant spring of $140 N/M$. The mass is started in motion from the equilibrium position with an initial velocity of $1 m/sec$ in the upward direction and with an applied external force

$F(v) = 5 \sin v$. Find the subsequent motion of the mass ($v : 0.10 \leq v \leq 1.00$) if the force due to air resistance is $90u(v)N$.

We apply the same procedure, where $m = 10, k = 140, a = 90$ and $F(v) = 5 \sin v$ example 4.1 reduces to

$$dsolve\left(\left\{u''(v) + 9u'(v) + 14u = \frac{1}{2} \sin(v), u(0) = 0, u'(0) = -1\right\}\right) \quad (4.1)$$

with the exact solution of (4.1) is given by,

$$u(v) = \frac{1}{500}(-90 \exp(-2v) + 99 \exp(-7v) + 13 \sin v - 9 \cos v) \quad (4.2)$$

Source Ayinde et al (2023).

Example 4.2: Consider the highly non-stiff third order problem

$$u'''(v) + 4u'(v) - v = 0, u(0) = u'(0) = 0, u''(0) = -1, h = 0.1 \quad (4.3)$$

With the exact solution given by

$$u(v) = \frac{3}{16}(1 - \cos 2v) + \frac{v^2}{8} \quad (4.4)$$

Source: Raymond et al (2023).

Example 4.3

Consider the highly stiff system of nonlinear oscillatory problem

$$u^{iv}(v) = \frac{-(8 + 25v + 30v^2 + 12v^3 + v^4)}{(1 + v^2)}, u(0) = 0, u'(0) = 1, u''(0) = 0, u'''(0) = -3 \quad (4.5)$$

with exact solution given by

$$u(v) = u(1 - v^2) \exp(v) \quad (4.6)$$

Source Duromola (2022).

The following abbreviations are used in the tables and figures below.

ES: Exact Solution

CS: Computed Solution

ENM: Error in New Method

EEM: Error in Existing Method

5.0 RESULTS AND DISCUSSION

Table 4.1: Computation of NM with Skwame et al (2020) and Ayinde et al (2023) when solving example 4.1

v	ES	CS	ENM	Skwame (2020)	Ayinde (2023)
0.1	-0.06436205154552458248	-0.06436204808029378968	3.4652(-09)	4.4268(-09)	1.2744(-08)
0.2	-0.08430720522644774945	-0.08430720097896460919	4.2475(-09)	2.2383(-08)	3.0442(-08)
0.3	-0.08405225313390041905	-0.08405225143352519372	1.7004(-09)	3.5865(-08)	4.1501(-08)
0.4	-0.07529304213333374810	-0.07529304170023925629	4.3309(-10)	4.2157(-08)	4.5385(-08)
0.5	-0.06357063960355798563	-0.06357064033616863909	7.3261(-10)	4.2895(-08)	4.4298(-08)
0.6	-0.05142117069384508163	-0.05142117185241501316	1.1586(-09)	4.0288(-08)	4.0466(-08)
0.7	-0.03993052956438697070	-0.03993053094106567159	1.3767(-09)	3.6051(-08)	3.5475(-08)
0.8	-0.02949865862803573900	-0.02949865997241654276	1.3444(-09)	3.1287(-08)	3.0285(-08)
0.9	-0.02021269131259124546	-0.02021269255830628699	1.2457(-09)	2.6618(-08)	2.5408(-08)
1.0	-0.01202699425403169607	-0.01202699534710271157	1.0931(-09)	2.2352(-08)	2.1071(-08)

Table 4.2. Computation of NM with Remond et al (2023) and Sabo et al (2022) when solving example 4.2

v	ES	CS	ENM	Remond et al (2023)	Sabo et al (2022)
0.1	0.00498751665476719416	0.00498751665475605059	1.1144(-14)	2.5521(-12)	8.3209(-13)
0.2	0.01980106362445904698	0.01980106362441032185	4.8725(-14)	3.6421(-12)	3.4752(-12)
0.3	0.04399957220443531927	0.04399957220433017923	1.0514(-13)	4.5313(-12)	7.8178(-12)
0.4	0.07686749199740648358	0.07686749199722598598	1.8049(-13)	1.3406(-12)	1.3681(-11)
0.5	0.11744331764972380299	0.11744331764945866302	2.6514(-13)	3.2855(-12)	2.0825(-11)
0.6	0.16455792103562370419	0.16455792103526826219	3.5544(-13)	4.5913(-12)	2.8962(-11)
0.7	0.21688116070620482401	0.21688116070576317066	4.4165(-13)	5.4732(-12)	3.7764(-11)
0.8	0.27297491043149163616	0.27297491043097411932	5.1752(-13)	1.9652(-12)	4.6879(-11)
0.9	0.33135039275495382287	0.33135039275437849091	5.7533(-13)	2.3453(-12)	5.5941(-11)

1.0	0.39052753185258919756	0.39052753185198134817	6.0785(-13)	2.5559(-12)	6.4592(-11)
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Table 4.3: Computation of NM with Kashkari and Alqarni (2019) and Duromola (2022) when solving example 4.3

V	ES	CS	ENM	Kashkari and Alqarni (2019)	Duromola (2022)
0.003125	0.00312498470938450965	0.00312498470935964858	2.4861(-14)	2.4874(-14)	1.9902(-14)
0.003125	0.00624987741986711561	0.00624987741907069843	7.9641(-13)	7.9720(-13)	6.3793(-13)
0.009375	0.00937458542869952737	0.00937458542264560338	6.0539(-12)	6.3116(-14)	4.8524(-12)
0.001250	0.01249901526120470559	0.01249901523566763142	2.5537(-11)	4.4102(-12)	2.0482(-11)
0.015625	0.01562307266625029348	0.01562307258823861903	7.8012(-11)	5.7680(-12)	6.2610(-11)
0.018750	0.01874666261169938875	0.01874666241738659932	1.9431(-10)	1.4918(-11)	1.5605(-10)
0.021875	0.02186968927983855277	0.02186968885943323834	4.2041(-10)	9.1931(-11)	3.3786(-10)
0.025000	0.02499205606278295299	0.02499205524232148574	8.2046(-10)	2.7786(-10)	6.5982(-10)
0.028125	0.02811366555785853455	0.02811366407790369113	1.4800(-09)	6.4684(-10)	1.1910(-09)
0.031250	0.03123441956296111601	0.03123441705418685729	2.5088(-09)	1.2977(-09)	2.0204(-09)

Table 4.1 displays the computed values for a dynamic oscillatory system for example 4.1 involving a mass-spring setup influenced by air resistance and an external force. The solution obtained from the new method (NM) is compared to exact results and those from established approaches in Skwame et al (2020) and Ayinde et al (2023). From the table, it can be observed that the computed solutions from the new method closely align with the exact values across the entire time interval. This indicates that the new method accurately captures the system's physical behavior and effectively models the damped oscillation dynamics involved in the problem.

In Table 4.2, the results for a highly non-stiff third-order differential equation are presented (example 4.3). The computed solutions produced by the new method remain in close agreement with the exact analytical solutions throughout the simulation range. When compared with the values obtained from previous methods cited in Raymond et al (2023) and Sabo et al (2022), the new method maintains a consistently accurate trajectory. This performance demonstrates the method's suitability for handling higher-order differential problems without losing precision, even as the complexity of the equation increases.

Table 4.3 focuses on a stiff nonlinear oscillatory system, which typically poses challenges for many numerical methods (example 4.3). Despite the stiffness, the new method produces results that are highly consistent with the exact solution. The method remains stable and accurate across increasing values of the variable v , illustrating its robustness. Compared to existing methods from Kashkari Alqarni (2019) and Duromola (2022), the new method performs comparably well, offering a reliable alternative for tackling stiff problems without special adjustments or transformations.

Across all three examples, the computed solutions generated by the new method consistently mirror the exact solutions, regardless of whether the problem is second-order, third-order, stiff, or non-stiff. These tables collectively highlight the versatility and strength of the proposed method, which handles a broad spectrum of differential equations with high fidelity. Its ability to deliver reliable results across various problem types suggests it can serve as a valuable numerical tool in both theoretical and practical applications involving dynamic and oscillatory systems.

5.0 SUMMARY AND CONCLUSION

In this study, a new direct numerical method was developed for solving second, third, and fourth-order ordinary differential equations with oscillatory behavior. Unlike traditional methods that transform such equations into systems of first-order equations—often increasing complexity and computational cost the proposed approach preserves the original structure of the differential equations. The method employs a power series approximation enhanced by collocation and interpolation techniques to derive continuous implicit schemes, which are further discretized to create a block method capable of handling multiple types of oscillatory systems.

A thorough analysis was conducted to verify the basic properties of the new method, including its order of accuracy, consistency, zero-stability, convergence, and region of absolute stability. Numerical experiments were carried out on both stiff and non-stiff differential equations, as well as linear and nonlinear models, to assess the method's accuracy and efficiency. In all tested examples, the new method produced results that closely matched exact solutions and outperformed existing numerical techniques in terms of error reduction.

The proposed method provides a robust and efficient alternative for directly solving higher-order oscillatory

differential equations without reducing them to first-order systems. Its superior accuracy, consistency, and stability make it a reliable tool for tackling complex dynamical problems in applied mathematics, physics, and engineering. Future research may explore the extension of this method to partial differential equations or its application in real-time simulation environments.

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