

A Study on Bivariate Exponentiated Nadarajah-Haghighi Distribution Derived from Plackett Copula

Hassan Idris^{1,2}, Sani Salihu Abubakar¹, Shamsuddeen Ahmad Sabo^{1*}

¹Department of Statistics, Aliko Dangote University of Science and Technology, Wudil-Nigeria

²Department of Statistics, Shehu Idris College of Health Science and Technology Makarfi-Nigeria

*Corresponding Author's Email & Phone No.: shamsuddeenahmadsabo@gmail.com ; +234-8067886262

Received on: April, 2025

Revised and Accepted on: May, 2025

Published on: July, 2025

ABSTRACT

The Nadarajah-Haghighi distribution is an essential lifetime distribution in survival analysis that serves as an alternative to the Weibull, beta and exponentiated exponential distributions. In this paper, a new bivariate distribution is proposed using the exponentiated Nadarajah-Haghighi distribution. The joint probability density function of the proposed distribution was established using Plackett copula function. The model's parameters were established using the Maximum Likelihood Estimation (MLE) procedure. An application using real datasets to depict the usefulness of the proposed model was demonstrated using two different datasets. The results of the applications indicated that the new bivariate distribution fit the real data and executed practically better than its competitors.

Keywords: Exponentiated Nadarajah-Haghighi Distribution, Bivariate models, Plackett Copula Function, Maximum Likelihood Estimation.

1.0 INTRODUCTION

Exponential distribution has a constant hazard rate, memory less property and a decreasing probability density function. Hence, it is holding in modeling real life data that indicated monotone and/ or non-monotone failure rate behaviors (Usman & Aliyu, 2022). To solve this problem, different researchers have extended the exponential distribution. For instance, the exponentiated Nadarajah-Haghighi distribution by (Singh *et al.*, 2023) and inverted Nadarajah-Haghighi distribution by (Ahsan-ul-Haq *et al.*, 2024). Nadarajah and Haghighi (2011) introduced a new distribution called Nadarajah-Haghighi distribution which serves as an extension to exponential distribution in order to increase its flexibility. Different researchers have introduced some extensions of the Nadarajah-Haghighi distribution some the notable ones include the: Marshal Olkin exponentiated Nadarajah-Haghighi distribution by (Sebastian *et al.*, 2024), Weibul inverse Nadarajah-Haghighi distribution by (Ahmad *et al.*, 2024), logistic Nadarajah-Haghighi distribution by (Pena-Ramirez *et al.*, 2020) and generalized Nadarajah-Haghighi distribution by (Ebere *et al.* 2024) among others.

All of these distributions are used in modelling univariate lifetime data. However, to model lifetime data of coupled observations such as eyes, kidneys, ears, double recurrence of a certain disease, familial association between various genetic diseases like breast cancer, diabetes and heart diseases, bivariate models are used. Some of these bivariate models include the bivariate Log-Top-Leone distribution (Usman *et al.*, 2023), the

bivariate Nadarajah-Haghighi (Usman & Aliyu, 2022; Aliyu & Usman, 2023) and the bivariate modified Rayleigh distribution (Hiba & Hagar, 2023). In this paper, a new bivariate distribution called bivariate exponentiated Nadarajah-Haghighi distribution is introduced as an extension of the exponentiated Nadarajah-Haghighi distribution using Plackett copula function, a widely used approach for modelling dependences in bivariate data as it provides a symmetric dependence, allowing for the modeling of both positive and negative correlations unlike some copulas that are poor on extreme dependence.

The rest of the paper is structured as follows, the model is proposed in section 2, some statistical properties of the proposed model including partial derivatives, survival and hazard functions are discussed in section 3 while model parameters were estimated via the maximum likelihood method of estimation in section 4. Real life applications are given in section 5, finally section 6 concludes the study and suggests potential directions for future research.

2.0 NEW BIVARIATE EXPONENTIATED NADARAJAH-HAGHIGHI DISTRIBUTION

This section presents the formulation of the new bivariate exponential Nadarajah-Haghighi distribution. It covers the introduction of the exponentiated Nadarajah-Haghighi distribution, the Plackett copula function, the partial derivative of the joint and marginal distributions, survival and hazard functions, and dependence measures using Plackett copula function.

2.1 Exponentiated Nadarajah-Haghighi Distribution

A new stimulus generalization of the exponential distribution which is named as the exponentiated Nadarajah-Haghighi distribution was achieved by adding

a duplicate parameter in Nadarajah-Haghighi distribution (see Singh *et al.*, 2023). The cumulative distribution function (*cdf*), probability density function (*pdf*), survival and hazard functions of the exponentiated Nadarajah-Haghighi distribution are respectively given by:

$$F(t) = \left[1 - \exp\left(1 - (1 + \lambda t)^\theta\right) \right]^\alpha \tag{1}$$

$$f(t) = \lambda \alpha \theta (1 + \lambda t)^{\theta-1} \exp\left(1 - (1 + \lambda t)^\theta\right) \left[1 - \exp\left(1 - (1 + \lambda t)^\theta\right) \right]^{\alpha-1} \tag{2}$$

$$S(t) = 1 - \left[1 - \exp\left(1 - (1 + \lambda t)^\theta\right) \right]^\alpha \tag{3}$$

And,

$$h(t) = \frac{\lambda \alpha \theta (1 + \lambda t)^{\theta-1} \exp\left(1 - (1 + \lambda t)^\theta\right) \left[1 - \exp\left(1 - (1 + \lambda t)^\theta\right) \right]^{\alpha-1}}{1 - \left[1 - \exp\left(1 - (1 + \lambda t)^\theta\right) \right]^\alpha} \tag{4}$$

Where; $\lambda > 0$ is a scale parameter and $\theta, \alpha > 0$ are shape parameters.

2.2 Plackett Copula Function

The word *copula* is derived from Latin, meaning to “connect together”. Copula functions are used to combine two or more univariate distributions in to bivariate or multivariate distributions. Copula function is a useful approach to describe a multivariate distribution with dependence structure. It is a multivariate probability distribution for which the marginal probability distribution of each variable is uniform on the interval (0, 1).

Consider two random variables T_1 and T_2 , each with distribution functions $F_1(t_1)$ and $F_2(t_2)$ respectively, each of $F_1(t_1)$ and $F_2(t_2)$ is uniformly distributed on the interval (0, 1). Aasen (2021) demonstrated that, there exists a copula C such that, for all T_1 and T_2 , there is a joint distribution function given by:

$$F(t_1, t_2) = C(F_1(t_1), F_2(t_2)) \tag{5}$$

Here, $F_1(t_1)$ and $F_2(t_2)$ are the cumulative distribution functions of the random variables T_1 and T_2 respectively. Several types of copula functions have been developed and studied by different researchers. Muhammad (2021) and Liu (2021) provided comprehensive reviews of various copula types. However, this study adopts the Plackett copula function to construct the bivariate exponentiated Nadarajah-Haghighi distribution.

The Plackett copula function is one of the most widely used parametric families of copula functions. Its cumulative distribution function is defined as:

$$F(t_1, t_2) = \frac{1 + (\gamma - 1)(F_1(t_1) + F_2(t_2)) - \sqrt{\left[1 + (\gamma - 1)(F_1(t_1) + F_2(t_2)) \right]^2 - 4F_1(t_1)F_2(t_2)(\gamma - 1)}}{2(\gamma - 1)} \tag{6}$$

Where; $\gamma \in (0, \infty)$. The correlation measure, Spearman’s rho is $\rho = \frac{\gamma + 1}{\gamma - 1} - \frac{2\gamma \log(\gamma)}{(\gamma - 1)^2}$. There is no closed expression in γ for the correlation measure Kendall’s tau.

2.3 The Proposed Model

Consider T_1 and T_2 be two lifetimes associated to the same individual with a dependence structure given by the Plackett copula function. Assume also, that T_1 and T_2 follows the exponentiated Nadarajah-Haghighi (*ENH*) distribution with respective parameters $\alpha_1, \theta_1, \lambda_1$ and $\alpha_2, \theta_2, \lambda_2$. That is, $T_1 \sim ENH(\alpha_1, \theta_1, \lambda_1)$ and $T_2 \sim ENH(\alpha_2, \theta_2, \lambda_2)$. Then, the cumulative distribution functions (*cdf*) for T_1 and T_2 are respectively given by:

$$F_1(t_1) = \left[1 - \exp\left(1 - (1 + \lambda_1 t_1)^{\theta_1}\right) \right]^{\alpha_1} \tag{7}$$

and

$$F_2(t_2) = \left[1 - \exp\left(1 - (1 + \lambda_2 t_2)^{\theta_2}\right) \right]^{\alpha_2} \tag{8}$$

The probability density function corresponding to the *cdfs* in equations (7) and (8) are respectively given by:

$$f_1(t_1) = \lambda_1 \theta_1 \alpha_1 (1 + \lambda_1 t_1)^{\theta_1 - 1} \exp\left(1 - (1 + \lambda_1 t_1)^{\theta_1}\right) \left[1 - \exp\left(1 - (1 + \lambda_1 t_1)^{\theta_1}\right) \right]^{\alpha_1 - 1} \tag{9}$$

and

$$f_2(t_2) = \lambda_2 \theta_2 \alpha_2 (1 + \lambda_2 t_2)^{\theta_2 - 1} \exp\left(1 - (1 + \lambda_2 t_2)^{\theta_2}\right) \left[1 - \exp\left(1 - (1 + \lambda_2 t_2)^{\theta_2}\right) \right]^{\alpha_2 - 1} \tag{10}$$

with survival functions given by $S_1(t_1) = 1 - F_1(t_1)$ and $S_2(t_2) = 1 - F_2(t_2)$ respectively. The joint *cdf* of the Plackett bivariate exponentiated Nadarajah-Haghighi distribution is obtained by substituting equations (7) and (8) into equation (6) which gives:

$$F(t_1, t_2) = \frac{1}{2(\gamma - 1)} \left[\begin{aligned} & 1 + (\gamma - 1) \left(\left[1 - \exp\left(1 - (1 + \lambda_1 t_1)^{\theta_1}\right) \right]^{\alpha_1} + \left[1 - \exp\left(1 - (1 + \lambda_2 t_2)^{\theta_2}\right) \right]^{\alpha_2} \right) \\ & - \sqrt{1 + (1 - \gamma) \left(\left[1 - \exp\left(1 - (1 + \lambda_1 t_1)^{\theta_1}\right) \right]^{\alpha_1} + \left[1 - \exp\left(1 - (1 + \lambda_2 t_2)^{\theta_2}\right) \right]^{\alpha_2} \right)^2} \\ & - \sqrt{-4(\gamma - 1) \left[1 - \exp\left(1 - (1 + \lambda_1 t_1)^{\theta_1}\right) \right]^{\alpha_1} \left[1 - \exp\left(1 - (1 + \lambda_2 t_2)^{\theta_2}\right) \right]^{\alpha_2}} \end{aligned} \right] \tag{11}$$

Hence, the joint *cdf* in equation (11) is the *cdf* of Bivariate Exponentiated Nadarajah-Haghighi distribution based on Plackett copula function. From now onward, this distribution will be denoted by BENH-Plackett distribution.

3.0 STATISTICAL PROPERTIES OF THE BENH-PLACKETT DISTRIBUTION

In this section, some statistical properties of the proposed BENH-Plackett distribution are studied, with a focus on its applicability in data analysis and survival studies.

3.1 Partial Derivatives of the BENH-Plackett Distribution

The partial derivatives w.r.t t_1 of the joint *cdf* of the BENH-Plackett distribution as given in equation (6), are obtained as follows:

$$\frac{\partial F(t_1, t_2)}{\partial t_1} = \frac{f_1(t_1) \left(1 - \left[(1 + (\gamma - 1) F_1(t_1) F_2(t_2)) - 4 F_2(t_2) \right] \right)}{2 \times \sqrt{1 + (\gamma - 1) \left(F_1(t_1) + F_2(t_2) \right)^2 - 4 F_1(t_1) F_2(t_2)}} \tag{12}$$

Hence, substituting equations (7), (8) and (9) into equation (12) yields the derivatives as follows;

$$\frac{\partial F(t_1, t_2)}{\partial t_1} = \left[\lambda_1 \theta_1 \alpha_1 (1 + \lambda_1 t_1)^{\theta_1 - 1} \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1 - 1} \right. \\ \left. \left(\left(1 - (1 + (\gamma - 1)) \left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1} \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2} \right) - 4 \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2} \right) \right. \\ \left. \div 2 \left[\sqrt{1 + (\gamma - 1)} \left(\left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1} + \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2} \right) \right. \right. \\ \left. \left. - 4 \left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1} \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2} \right] \right] \quad (13)$$

While the partial derivative with respect to t_2 is obtained as:

$$\frac{\partial F(t_1, t_2)}{\partial t_2} = \frac{f_2(t_2) (1 - [(1 + (\gamma - 1)) F_1(t_1) F_2(t_2)] - 4 F_1(t_1))}{2 \times \sqrt{1 + (\gamma - 1)} [(F_1(t_1) + F_2(t_2))^2 - 4 F_1(t_1) F_2(t_2)]} \quad (14)$$

Hence, substituting equations (7), (8) and

(10) into equation (14) yields the first derivative as follows:

$$\frac{\partial F(t_1, t_2)}{\partial t_2} = \left[\lambda_2 \theta_2 \alpha_2 (1 + \lambda_2 t_2)^{\theta_2 - 1} \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2 - 1} \right. \\ \left. \left(\left(1 - (1 + (\gamma - 1)) \left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1} \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2} \right) - 4 \left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1} \right) \right. \\ \left. \div 2 \left[\sqrt{1 + (\gamma - 1)} \left(\left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1} + \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2} \right) \right. \right. \\ \left. \left. - 4 \left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1} \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2} \right] \right] \quad (15)$$

The second partial derivative of the BENH-Plackett is obtained by either differentiating (12) partially with respect to t_2 or differentiating (14) partially with respect to t_1 which yields:

$$\frac{\partial^2 F(t_1, t_2)}{\partial t_1 \partial t_2} = \frac{\gamma f_1(t_1) f_2(t_2) [1 + (F_1(t_1) - 2 F_1(t_1) F_2(t_2) + F_2(t_2)) (\gamma - 1)]}{[1 + (\gamma - 1) (F_1(t_1) + F_2(t_2))^2 - 4 \gamma (\gamma - 1) F_1(t_1) F_2(t_2)]^{\frac{3}{2}}} \quad (16)$$

Hence, substituting equations (7), (8), (9) and (10) in equation (16) gives the second derivative with respect to:

$$\frac{\partial^2 F(t_1, t_2)}{\partial t_1 \partial t_2} = \left[\lambda_1 \theta_1 \alpha_1 (1 + \lambda_1 t_1)^{\theta_1 - 1} \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1 - 1} \lambda_2 \theta_2 \alpha_2 (1 + \lambda_2 t_2)^{\theta_2 - 1} \right. \\ \times \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2 - 1} \left[1 + \left(\left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1} - \right. \right. \\ \left. \left. \times 2 \left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1} \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2} + \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2} \right) (\gamma - 1) \right] \div \\ \left[\left(1 + (\gamma - 1) \left(\left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1} + \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2} \right) \right)^2 - \right. \\ \left. 4 \gamma \left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1} \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2} \right]^{\frac{3}{2}} \\ = f(t_1, t_2) \quad (17)$$

The second derivative in equation (17) is the joint *pdf* of T_1 and T_2 for the BENH-Plackett distribution, which is

$$f(t_1, t_2) = \frac{\partial^2 F(t_1, t_2)}{\partial t_1 \partial t_2}.$$

3.2 Survival Function of the BENH-Plackett Distribution

The survival function of the *BENH-Plackett* distribution can easily be obtained using the relation:

$$S(t_1, t_2) = \frac{1 + (\gamma - 1)((1 - F_1(t_1)) + (1 - F_2(t_2)))\sqrt{[1 + (\gamma - 1)((1 - F_1(t_1)) + (1 - F_2(t_2)))^2] - 4(1 - F_1(t_1))(1 - F_2(t_2))(\gamma - 1)}}{2(\gamma - 1)} \tag{18}$$

The Plackett copula function is obtained by substituting equations (7) and (8) into equation (18), which is given as:

$$S(t_1, t_2) = \frac{1}{2(\gamma - 1)} \left[\frac{1 + (\gamma - 1)\left(\left(1 - \left[1 - \exp\left(1 - (1 + \lambda_1 t_1)^{\theta_1}\right)\right]^{\alpha_1}\right) + \left(1 - \left[1 - \exp\left(1 - (1 + \lambda_2 t_2)^{\theta_2}\right)\right]^{\alpha_2}\right)\right)}{\sqrt{1 + (1 - \gamma)\left(\left(1 - \left[1 - \exp\left(1 - (1 + \lambda_1 t_1)^{\theta_1}\right)\right]^{\alpha_1}\right) + \left(1 - \left[1 - \exp\left(1 - (1 + \lambda_2 t_2)^{\theta_2}\right)\right]^{\alpha_2}\right)\right)^2 - 4(\gamma - 1)\left(1 - \left[1 - \exp\left(1 - (1 + \lambda_1 t_1)^{\theta_1}\right)\right]^{\alpha_1}\right)\left(1 - \left[1 - \exp\left(1 - (1 + \lambda_2 t_2)^{\theta_2}\right)\right]^{\alpha_2}\right)}} \right] \tag{19}$$

The survival function of the *BENH-Plackett* distribution in equation (19) tends to one (1) as $t_1, t_2 \rightarrow 0$, and it tends to zero (0) as $t_1, t_2 \rightarrow \infty$. This is given as follows:

$$\lim_{t_1 \rightarrow 0} \lim_{t_2 \rightarrow 0} S(t_1, t_2) = \frac{1}{2(\gamma - 1)} \left[\frac{1 + (\gamma - 1)\left(\left(1 - \left[1 - \exp\left(1 - (1 + \lambda_1 t_1)^{\theta_1}\right)\right]^{\alpha_1}\right) + \left(1 - \left[1 - \exp\left(1 - (1 + \lambda_2 t_2)^{\theta_2}\right)\right]^{\alpha_2}\right)\right)}{\sqrt{1 + (1 - \gamma)\left(\left(1 - \left[1 - \exp\left(1 - (1 + \lambda_1 t_1)^{\theta_1}\right)\right]^{\alpha_1}\right) + \left(1 - \left[1 - \exp\left(1 - (1 + \lambda_2 t_2)^{\theta_2}\right)\right]^{\alpha_2}\right)\right)^2 - 4(\gamma - 1)\left(1 - \left[1 - \exp\left(1 - (1 + \lambda_1 t_1)^{\theta_1}\right)\right]^{\alpha_1}\right)\left(1 - \left[1 - \exp\left(1 - (1 + \lambda_2 t_2)^{\theta_2}\right)\right]^{\alpha_2}\right)}} \right]$$

Which further simplifies to;

$$\lim_{t_1 \rightarrow 0} \lim_{t_2 \rightarrow 0} S(t_1, t_2) = \frac{1 + 2(\gamma - 1) - \sqrt{1 + 4(\gamma - 1) - 4(\gamma - 1)}}{2(\gamma - 1)} = 1 \tag{20}$$

Also;

$$\lim_{t_1 \rightarrow \infty} \lim_{t_2 \rightarrow \infty} S(t_1, t_2) = \frac{1}{2(\gamma - 1)} \left[\frac{1 + (\gamma - 1)\left(\left(1 - \left[1 - \exp\left(1 - (1 + \lambda_1 t_1)^{\theta_1}\right)\right]^{\alpha_1}\right) + \left(1 - \left[1 - \exp\left(1 - (1 + \lambda_2 t_2)^{\theta_2}\right)\right]^{\alpha_2}\right)\right)}{\sqrt{1 + (1 - \gamma)\left(\left(1 - \left[1 - \exp\left(1 - (1 + \lambda_1 t_1)^{\theta_1}\right)\right]^{\alpha_1}\right) + \left(1 - \left[1 - \exp\left(1 - (1 + \lambda_2 t_2)^{\theta_2}\right)\right]^{\alpha_2}\right)\right)^2 - 4(\gamma - 1)\left(1 - \left[1 - \exp\left(1 - (1 + \lambda_1 t_1)^{\theta_1}\right)\right]^{\alpha_1}\right)\left(1 - \left[1 - \exp\left(1 - (1 + \lambda_2 t_2)^{\theta_2}\right)\right]^{\alpha_2}\right)}} \right] \tag{22}$$

Which simplifies to;

$$\lim_{t_1 \rightarrow \infty} \lim_{t_2 \rightarrow \infty} S(t_1, t_2) = \frac{1}{2(\gamma - 1)} [1 - 1] = 0 \tag{23}$$

3.3 Bivariate Failure Rate Function of the BENH-Plackett Distribution

The hazard function of the *BENH-Plackett* distribution can easily be obtained using the relation:

$$h(t_1, t_2) = \frac{f(t_1, t_2)}{S(t_1, t_2)} \tag{24}$$

The hazard function of the *BENH*-Plackett distribution is obtained by substituting equations (17) and (19) into the equation (24), which is given as:

$$h(t_1, t_2) = \left[\frac{\gamma \lambda_1 \theta_1 \alpha_1 (1 + \lambda_1 t_1)^{\theta_1 - 1} \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1 - 1} \lambda_2 \theta_2 \alpha_2 (1 + \lambda_2 t_2)^{\theta_2 - 1} \times \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2 - 1} \left[1 + \left(\left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1} - 2 \left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1} \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2} + \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2} \right) (\gamma - 1) \right]}{\left(\left[1 + (\gamma - 1) \left(\left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1} + \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2} \right) \right]^2 - 4\gamma \left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1} \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2} \right)^{\frac{3}{2}}} \right] \div \left[\frac{1}{2(\gamma - 1)} \left[\frac{1 + (\gamma - 1) \left(\left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1} + \left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2} \right)}{\sqrt{-4(\gamma - 1) \left(\left[1 - \exp(1 - (1 + \lambda_1 t_1)^{\theta_1}) \right]^{\alpha_1} \right) \left(\left[1 - \exp(1 - (1 + \lambda_2 t_2)^{\theta_2}) \right]^{\alpha_2} \right)}} \right] \right] \tag{25}$$

4.0 ESTIMATION AND DISCRIMINATION PROCEDURES

This section explains how the parameters of the proposed Bivariate Exponentiated Nadarajah-Haghighi (*BENH*) Plackett distribution were estimated using the Maximum Likelihood Estimation (MLE) method. In addition to the parameter estimation, the discrimination procedures are also presented.

4.1 Estimation Procedure

The parameters of the proposed Bivariate Exponentiated Nadarajah-Haghighi (*BENH*) Plackett distribution are estimated using the Maximum Likelihood Estimation (MLE) procedure. MLE is one of the most popular and orderly used method in statistics for estimating the parameters of a statistical model. For more detailed informations, reader can refer to Osi *et al.*, (2024).

Let $t_{11}, t_{12}, \dots, t_{1n}$, and $t_{21}, t_{22}, \dots, t_{2n}$ be random samples from *BENH* distribution, then the likelihood function is given as:

$$L(\Phi / t_{11}, t_{12}, \dots, t_{1n}, t_{21}, t_{22}, \dots, t_{2n}) = \prod_{i=1}^n f(t_{1i}, t_{2i}) \tag{26}$$

Here, Φ is a vector of parameters. That is, $\Phi = (\lambda_1, \lambda_2, \theta_1, \theta_2, \alpha_1, \alpha_2, \gamma)$

The likelihood function for the *BENH*-Plackett distribution is obtained by substituting the joint *pdf* in equation (17) into equation (26).

Let $A_1 = 1 - \exp(1 - (1 + \lambda_1 t_{1i})^{\theta_1})$ and $A_2 = 1 - \exp(1 - (1 + \lambda_2 t_{2i})^{\theta_2})$

$$L(\Phi) = \frac{\alpha_1^n \alpha_2^n \lambda_1^n \lambda_2^n \theta_1^n \theta_2^n \gamma^n \exp\left(\sum_{i=1}^n 2 - (1 + \lambda_1 t_{1i})^{\theta_1} - (1 + \lambda_2 t_{2i})^{\theta_2}\right) \prod_{i=1}^n A_1^{\alpha_1 - 1} A_2^{\alpha_2 - 1} \left(1 + (A_1^{\alpha_1} - 2A_2^{\alpha_2} + A_2^{\alpha_2}) (\gamma - 1)\right)}{\prod_{i=1}^n \left[1 + (\gamma - 1)(A_1^{\alpha_1} + A_2^{\alpha_2})\right]^2 - 4\gamma(\gamma - 1)A_1^{\alpha_1} A_2^{\alpha_2}} \quad (27)$$

Taking the natural logarithm of both sides of equation (27) yields:

$$l(\Phi) = n \ln \alpha_1 + n \ln \alpha_2 + n \ln \theta_1 + n \ln \theta_2 + n \ln \lambda_1 + n \ln \lambda_2 + n \ln \gamma + n \ln(\gamma - 1) + \ln(n) + 2n - \sum_{i=1}^n (1 + \lambda_1 t_{1i})^{\theta_1} - \sum_{i=1}^n (1 + \lambda_2 t_{2i})^{\theta_2} + (\theta_1 - 1) \sum_{i=1}^n (1 + \lambda_1 t_{1i}) + (\theta_2 - 1) \sum_{i=1}^n (1 + \lambda_2 t_{2i}) + (\alpha_1 + 1) \sum_{i=1}^n \ln A_1 + (\alpha_2 + 1) \sum_{i=1}^n \ln A_2 + \sum_{i=1}^n \ln \left[1 + (A_1^{\alpha_1} - 2A_1^{\alpha_1} A_2^{\alpha_2} + A_1^{\alpha_1}) (\gamma - 1)\right] - \frac{3}{2} \sum_{i=1}^n \ln \left[1 + (\gamma - 1)(A_1^{\alpha_1} + A_2^{\alpha_2})\right]^2 - 4\gamma(\gamma - 1)A_1^{\alpha_1} A_2^{\alpha_2} \quad (28)$$

To obtain the Maximum Likelihood estimators (MLEs) of the parameters, λ_1 , λ_2 , θ_1 , θ_2 , α_1 , α_2 , and γ equation (28) should be partially differentiated with respect to each of the aforementioned parameters. These partial derivatives are obtained as follows:

$$\frac{\partial l(\Phi)}{\partial \alpha_1} = \frac{n}{\alpha_1} + \sum_{i=1}^n \ln(A_1) + \sum_{i=1}^n \frac{\alpha_1 \theta_1 (\theta_1 - 1) (1 + \lambda_1 t_{1i})^{\theta_1 - 1} A_1^{\alpha_1 - 1} (1 - A_1) (\gamma - 1)}{\left[1 + (A_1^{\alpha_1} - 2A_1^{\alpha_1} A_2^{\alpha_2} + A_1^{\alpha_1}) (\gamma - 1)\right]} - \sum_{i=1}^n \frac{2\alpha_1 \theta_1 (\theta_1 - 1) (1 + \lambda_1 t_{1i})^{\theta_1 - 1} A_1^{\alpha_1 - 1} A_2^{\alpha_2} (1 - A_1) (\gamma - 1)}{\left[1 + (A_1^{\alpha_1} - 2A_1^{\alpha_1} A_2^{\alpha_2} + A_1^{\alpha_1}) (\gamma - 1)\right]} - 3 \sum_{i=1}^n \frac{\left[1 + (\gamma - 1)(A_1^{\alpha_1} + A_2^{\alpha_2})\right] (\gamma - 1) \alpha_1 \theta_1 (1 + \lambda_1 t_{1i})^{\theta_1 - 1} (1 - A_1) - 4\gamma(\gamma - 1) \alpha_1 \theta_1 (1 + \lambda_1 t_{1i})^{\theta_1 - 1} A_1^{\alpha_1 - 1} (1 - A_1)}{\left[1 + (\gamma - 1)(A_1^{\alpha_1} + A_2^{\alpha_2})\right]^2 - 4\gamma(\gamma - 1)A_1^{\alpha_1} A_2^{\alpha_2}} \quad (29)$$

$$\frac{\partial l(\Phi)}{\partial \alpha_2} = \frac{n}{\alpha_2} + \sum_{i=1}^n \ln(A_2) + \sum_{i=1}^n \frac{\alpha_2 \theta_2 (\theta_2 - 1) (1 + \lambda_2 t_{2i})^{\theta_2 - 1} A_2^{\alpha_2 - 1} (1 - A_2) (\gamma - 1)}{\left[1 + (A_1^{\alpha_1} - 2A_1^{\alpha_1} A_2^{\alpha_2} + A_1^{\alpha_1}) (\gamma - 1)\right]} - \sum_{i=1}^n \frac{2\alpha_2 \theta_2 (\theta_2 - 1) (1 + \lambda_2 t_{2i})^{\theta_2 - 1} A_2^{\alpha_2 - 1} A_1^{\alpha_1} (1 - A_2) (\gamma - 1)}{\left[1 + (A_1^{\alpha_1} - 2A_1^{\alpha_1} A_2^{\alpha_2} + A_1^{\alpha_1}) (\gamma - 1)\right]} - 3 \sum_{i=1}^n \frac{\left[1 + (\gamma - 1)(A_1^{\alpha_1} + A_2^{\alpha_2})\right] (\gamma - 1) \alpha_2 \theta_2 (1 + \lambda_2 t_{2i})^{\theta_2 - 1} (1 - A_2) - 4\gamma(\gamma - 1) \alpha_2 \theta_2 (1 + \lambda_2 t_{2i})^{\theta_2 - 1} A_2^{\alpha_2 - 1} (1 - A_2)}{\left[1 + (\gamma - 1)(A_1^{\alpha_1} + A_2^{\alpha_2})\right]^2 - 4\gamma(\gamma - 1)A_1^{\alpha_1} A_2^{\alpha_2}} \quad (30)$$

$$\frac{\partial l(\Phi)}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^n \ln(A_1^{\alpha_1} - 2A_1^{\alpha_1} A_2^{\alpha_2} + A_1^{\alpha_1}) - 3 \sum_{i=1}^n \frac{\left[1 + (\gamma - 1)(A_1^{\alpha_1} + A_2^{\alpha_2})\right] (A_1^{\alpha_1} + A_2^{\alpha_2}) - 4A_1^{\alpha_1} A_2^{\alpha_2} (2\gamma - 1)}{\left[1 + (\gamma - 1)(A_1^{\alpha_1} + A_2^{\alpha_2})\right]^2 - 4\gamma(\gamma - 1)A_1^{\alpha_1} A_2^{\alpha_2}} \quad (31)$$

$$\begin{aligned} \frac{\partial l(\Phi)}{\partial \lambda_1} &= \frac{n}{\lambda_1} + (\theta_1 - 1) \sum_{i=1}^n \frac{\theta_1 (1 + \lambda_1 t_{1i})^{\theta_1 - 1} t_{1i}}{(1 + \lambda_1 t_{1i})^{\theta_1}} + (\alpha_2 - 1) \sum_{i=1}^n \frac{\theta_1 (1 + \lambda_1 t_{1i})^{\theta_1 - 1} (1 - A_1) t_{1i}}{A_1} \\ &+ \sum_{i=1}^n \frac{\alpha_1 \theta_1 (1 + \lambda_1 t_{1i})^{\theta_1 - 1} t_{1i} A_1^{\alpha_1 - 1} (1 - A_1) (\gamma - 1)}{1 + (A_1^{\alpha_1} - 2A_1^{\alpha_1} A_2^{\alpha_2} + A_1^{\alpha_1}) (\gamma - 1)} - \sum_{i=1}^n \frac{2\alpha_1 \theta_1 (1 + \lambda_1 t_{1i})^{\theta_1 - 1} t_{1i} A_1^{\alpha_1 - 1} A_2^{\alpha_2} (1 - A_1) (\gamma - 1)}{1 + (A_1^{\alpha_1} - 2A_1^{\alpha_1} A_2^{\alpha_2} + A_1^{\alpha_1}) (\gamma - 1)} \\ &- 3 \sum_{i=1}^n \frac{[1 + (\gamma - 1)(A_1^{\alpha_1} + A_2^{\alpha_2})] (\gamma - 1) \theta_1 (1 + \lambda_1 t_{1i}) t_{1i} (1 - A_1) - 4\gamma (\gamma - 1) \lambda_1 \theta_1 (1 + \lambda_1 t_{1i})^{\theta_1 - 1} t_{1i} A_1^{\alpha_1 - 1}}{[1 + (\gamma - 1)(A_1^{\alpha_1} + A_2^{\alpha_2})]^2 - 4\gamma (\gamma - 1) A_1^{\alpha_1} A_2^{\alpha_2}} \end{aligned} \tag{32}$$

$$\begin{aligned} \frac{\partial l(\Phi)}{\partial \lambda_2} &= \frac{n}{\lambda_2} + (\theta_2 - 1) \sum_{i=1}^n \frac{\theta_2 (1 + \lambda_2 t_{2i})^{\theta_2 - 1} t_{2i}}{(1 + \lambda_2 t_{2i})^{\theta_2}} + (\alpha_2 - 1) \sum_{i=1}^n \frac{\theta_2 (1 + \lambda_2 t_{2i})^{\theta_2 - 1} (1 - A_2) t_{2i}}{A_2} \\ &+ \sum_{i=1}^n \frac{\alpha_2 \theta_2 (1 + \lambda_2 t_{2i})^{\theta_2 - 1} t_{2i} A_2^{\alpha_2 - 1} (1 - A_2) (\gamma - 1)}{1 + (A_1^{\alpha_1} - 2A_1^{\alpha_1} A_2^{\alpha_2} + A_1^{\alpha_1}) (\gamma - 1)} - \sum_{i=1}^n \frac{2\alpha_2 \theta_2 (1 + \lambda_2 t_{2i})^{\theta_2 - 1} t_{2i} A_2^{\alpha_2 - 1} A_1^{\alpha_1} (1 - A_2) (\gamma - 1)}{1 + (A_1^{\alpha_1} - 2A_1^{\alpha_1} A_2^{\alpha_2} + A_1^{\alpha_1}) (\gamma - 1)} \\ &- 3 \sum_{i=1}^n \frac{[1 + (\gamma - 1)(A_1^{\alpha_1} + A_2^{\alpha_2})] (\gamma - 1) \theta_1 (1 + \lambda_1 t_{1i}) t_{1i} (1 - A_1) - 4\gamma (\gamma - 1) \lambda_1 \theta_1 (1 + \lambda_1 t_{1i})^{\theta_1 - 1} t_{1i} A_1^{\alpha_1 - 1}}{[1 + (\gamma - 1)(A_1^{\alpha_1} + A_2^{\alpha_2})]^2 - 4\gamma (\gamma - 1) A_1^{\alpha_1} A_2^{\alpha_2}} \end{aligned} \tag{33}$$

$$\begin{aligned} \frac{\partial l(\Phi)}{\partial \theta_1} &= \frac{n}{\theta_1} + \sum_{i=1}^n \ln(1 + \lambda_1 t_{1i}) - \sum_{i=1}^n (1 + \lambda_1 t_{1i})^{\theta_1} \ln(1 + \lambda_1 t_{1i}) + (\alpha_1 - 1) \sum_{i=1}^n \frac{\theta_1 (1 + \lambda_1 t_{1i})^{\theta_1 - 1} (1 - A_1)}{A_1} - \\ &\sum_{i=1}^n \frac{[\alpha_1 \theta_1 (1 + \lambda_1 t_{1i})^{\theta_1 - 1} A_1^{\alpha_1 - 1} (1 - A_1) - 2\alpha_1 \theta_1 (1 + \lambda_1 t_{1i})^{\theta_1 - 1} A_1^{\alpha_1 - 1} A_2^{\alpha_2} (1 - A_1)] (\gamma - 1)}{1 + (A_1^{\alpha_1} - 2A_1^{\alpha_1} A_2^{\alpha_2} + A_1^{\alpha_1}) (\gamma - 1)} \\ &- 3 \sum_{i=1}^n \frac{[1 + (\gamma - 1)(A_1^{\alpha_1} + A_2^{\alpha_2})] (\gamma - 1) \theta_1 (1 + \lambda_1 t_{1i})^{\theta_1 - 1} (1 - A_1) - 4\gamma (\gamma - 1) \alpha_1 \theta_1 (1 + \lambda_1 t_{1i})^{\theta_1 - 1} A_1^{\alpha_1 - 1} (1 - A_1)}{[1 + (\gamma - 1)(A_1^{\alpha_1} + A_2^{\alpha_2})]^2 - 4\gamma (\gamma - 1) A_1^{\alpha_1} A_2^{\alpha_2}} \end{aligned} \tag{34}$$

$$\begin{aligned} \frac{\partial l(\Phi)}{\partial \theta_2} = & \frac{n}{\theta_2} + \sum_{i=1}^n \ln(1 + \lambda_2 t_{2i}) - \sum_{i=1}^n (1 + \lambda_2 t_{2i})^{\theta_2} \ln(1 + \lambda_2 t_{2i}) + (\alpha_1 - 1) \sum_{i=1}^n \frac{\theta_2 (1 + \lambda_2 t_{2i})^{\theta_2 - 1} (1 - A_2)}{A_2} - \\ & \sum_{i=1}^n \frac{\left[\alpha_2 \theta_2 (1 + \lambda_2 t_{2i})^{\theta_2 - 1} A_2^{\alpha_2 - 1} (1 - A_2) - 2 \alpha_2 \theta_2 (1 + \lambda_2 t_{2i})^{\theta_2 - 1} A_2^{\alpha_2 - 1} A_1^{\alpha_1} (1 - A_2) \right] (\gamma - 1)}{1 + (A_1^{\alpha_1} - 2 A_1^{\alpha_1} A_2^{\alpha_2} + A_1^{\alpha_1}) (\gamma - 1)} \\ & - 3 \sum_{i=1}^n \frac{\left[1 + (\gamma - 1) (A_1^{\alpha_1} + A_2^{\alpha_2}) \right] (\gamma - 1) \theta_2 (1 + \lambda_2 t_{2i})^{\theta_2 - 1} (1 - A_2) - 4 \gamma (\gamma - 1) \alpha_2 \theta_2 (1 + \lambda_2 t_{2i})^{\theta_2 - 1} A_2^{\alpha_2 - 1} (1 - A_2)}{\left[1 + (\gamma - 1) (A_1^{\alpha_1} + A_2^{\alpha_2}) \right]^2 - 4 \gamma (\gamma - 1) A_1^{\alpha_1} A_2^{\alpha_2}} \end{aligned} \tag{35}$$

Equations (29) to (35) are non-linear; therefore, numerical techniques such as Newton-Raphson algorithm can be used to obtain estimates of the parameters.

4.2 Discrimination Procedure

To discriminate between the fits of the models obtained using classical estimation procedure, various goodness of fit measures are considered. These includes information criteria such as the Akaike Information Criterion (AIC), the Corrected Akaike Information Criterion (CAIC) and the Bayesian Information Criterion (BIC). These criteria are based on the log-likelihood function $l(\theta)$ of the model. According to these criteria, the smaller the value, the better the model fit. Let p denote the number of unknown parameter in the fitted model, and n represent the sample size of the data.

1. The Akaike Information Criterion (AIC) value is defined as:

$$AIC = -2 \ln L + 2\rho \tag{36}$$

2. The Corrected Akaike Information Criterion (CAIC) value is given by the:

$$CAIC = -2 \ln L + 2\rho \frac{n}{n - \rho - 1} \tag{37}$$

The CAIC represents the AIC with a correction for a small sample size n .

3. The Bayesian Information Criterion (BIC) value is computed as:

$$BIC = -2 \ln L + \rho \ln(n) \tag{38}$$

For each of the three criteria, a model with the smallest value is considered to have the best fit. For more details, see Osi *et al.*, (2025).

5.0 APPLICATION TO REAL DATA

In this section, the applicability of the established bivariate distribution is demonstrated using two datasets. The first dataset comprises the recurrence times of infections in 38 kidney patients, while the second dataset consist of the scores of 14 post-graduate students from the 2021/2022 academic session in the *Probability and Inference* course at Ahmadu Bello University Zaria, Nigeria. More detailed information about each dataset is provided in the subsequent sections.

5.1 Kidney Data

The first dataset used in demonstrating the applicability of the proposed BENH-Plackett distribution is the infections in kidney patients' dataset. This data was previously used by (Usman & Aliyu, 2022) and (Aliyu & Usman, 2023). The data consists of the recurrence times to infection at point of insertion of catheter using portable dialysis equipment for thirty-eight (38) kidney patients. Two recurrence times were recorded for each patient together with risk variable values (age, sex: male = 0, female = 1 and disease type). Let T_1 and T_2 be the first and second recurrence time of the infection kidney respectively. The recurrence times together with the aforementioned variables can be obtained in (Aliyu & Usman, 2023). This dataset was used in fitting the BENH-Plackett distribution and compare its performance with that of Bivariate Exponential (BE), Bivariate Nadarajah-Haghighi (BNH) and Bivariate Generalized Nadarajah-Haghighi (BGNH) distributions. The data is presented in Table A1 in appendix and the results of these fits are given in Table 1.

Table 1: Maximum Likelihood Estimates; Kidney infection data

Model	Parameter	Estimate	S. E	P	AIC	BIC	CAIC
Bivariate Exponentiated Nadarajah Haghighi	α_1	2.4540	0.6405	0.0001	478.7558	517.0832	482.0499
	α_2	6.3394	3.1385	0.0434			
	λ_1	0.6181	0.1810	0.0006			
	λ_2	4.5814	3.3561	0.1722			
	θ_1	11.5406	2.6963	<0.0000			
	θ_2	0.6304	0.0643	<0.0000			
	γ	0.3526	0.0396	<0.0000			
Bivariate Nadarajah Haghighi	λ_1	0.0230	0.0063	0.0002	527.8266	536.5149	521.8266
	λ_2	0.0083	0.0038	0.0291			
	θ_1	3.8554	3.3132	0.2446			
	θ_2	1.0497	0.2110	<0.0000			
	γ	0.5205	0.9538	0.0083			
Bivariate Exponential	λ_1	0.0282	0.0080	0.0003	515.8912	509.8912	511.8912
	λ_2	0.0278	0.0077	0.00023			
	γ	0.7310	2.4000	0.3357			
Bivariate Generalizd Exponential	λ_1	0.0520	0.0123	<0.0000	517.1810	507.1810	511.1801
	λ_2	0.0316	0.0107	0.0032			
	α_1	3.4507	6.2129	0.5786			
	α_2	4.1988	2.9030	0.14807			
	γ	0.9855	0.7364	0.0377			

Table 1 presents the maximum likelihood estimates of the parameters, along with the standard errors (S.E), p-values, AIC, BIC and CAIC. The results indicate that, the BENH distribution has the lowest values of AIC, BIC and CAIC. For example, the AIC value of BENH Plackett distribution is 478.7558 which is lower than the AIC value of 515.8912 and 517.1810 for the BE and BGE distributions, respectively.

5.2 Post Graduate Students Scores Data

The second dataset used in this work is a real data which is the score of 14 first-graduate students 2021/2022 academic session in Probability and Inference of Ahmadu Bello University in Nigeria. The dataset is given in Table A2 in the appendix and the results of the dataset in Table 2.

Table 2: Maximum Likelihood Estimates Result for Undergraduate Students Scores

Model	Parameter	Estimate	S. E	P	AIC	BIC	CAIC
Bivariate Exponentiated Nadarajah Haghighi	α_1	6.8410	2.9333	0.0197	254.5822	259.0556	273.2489
	α_2	4.8116	2.0177	0.0171			
	λ_1	0.0393	0.0126	0.0019			
	λ_2	0.0458	0.0197	0.0120			
	θ_1	0.5914	0.0193	0.0003			
	θ_2	1.0611	0.1999	<0.0000			
	γ	0.8971	0.2118	<0.0000			
	λ_1	0.1957	0.0116	<0.0000	267.8710	271.0663	261.8710



Bivariate Nadarajah Haghghi	λ_2	0.5042	0.0401	<0.0000			
	θ_1	1.5910	0.0062	<0.0000			
	θ_2	0.37920	0.0107	<0.0000			
	γ	0.30352	0.0092	<0.0000			
Bivariate Exponential	λ_1	0.0304	0.0017	<0.0000	262.2004	264.1175	258.2004
	λ_2	0.0527	0.0046	<0.0000			
	γ	1.5408	0.0052	<0.0000			
Bivariate Generalized Exponential	λ_1	0.0059	0.0065	<0.0000	257.7705	260.9657	251.7705
	λ_2	0.0272	0.0343	<0.0000			
	α_1	1.6800	0.0058	<0.0000			
	α_2	0.3747	0.0094	<0.0000			
	γ	1.2651	0.0543	<0.0000			

Table 2 presents the maximum likelihood estimates of the parameters along with the standard errors (S.E), p-values, AIC, BIC and CAIC. The results indicate that, the BENH distribution has the lowest values of AIC, BIC and CAIC. For example, the AIC value of BENH distribution is 254.5822 which is lower than the AIC value of 257.7705 for the BGE distribution.

6.0 CONCLUSION

In this paper, we proposed a novel bivariate lifetime distribution based on the exponentiated Nadarajah-Haghghi distribution, constructed using the Plackett copula function. We provided a detailed derivation of key statistical properties, including partial derivatives, survival function and failure rate function. The parameters of the proposed distribution were estimated using the maximum likelihood method. To assess the efficacy of the model, we applied it to two real-world datasets. The results evaluated using various goodness-of-fit criteria, demonstrate that the proposed model provides a superior fit compared to some existing competing models, highlighting its potential for practical applications in reliability and survival analyses. Given the promising performance of the proposed distribution, we recommended its application over other competing distributions. However, the proposed distribution can only handle bivariate data. Future research could explore the extension of the proposed distribution to higher dimensional cases, Bayesian estimation techniques, alternative dependence structures, etc.

ACKNOWLEDGEMENTS

The authors acknowledge the contribution of the reviewers and the editorial team for their insightful comments, which improves the quality of the paper.

REFERENCES

Aasen, N. R. (2021). An Introduction to Copula Theory (Bachelor's thesis, NTNU).

Aliyu, Y., & Usman, U. (2023). On Bivariate Nadarajah-Haghghi Distribution derived from Farlie-Gumbel-Morgenstern Copula in the Presence of Covariates. *Journal of the Nigerian Society of Physical Sciences*, 5, 871-871. DOI:10.46481/jnsps.2023.871

Ahmad, A., Alsadat, N., Rather, A. A., Meraou, M. A., & El-Din, M. M. M. (2024). A novel statistical approach to COVID-19 variability using the Weibull-Inverse Nadarajah Haghghi distribution. *Alexandria Engineering Journal*, 107, 950-962. DOI: <https://doi.org/10.1016/j.aej.2024.08.008>

Ahsan-ul-Haq, M., Shahzad, M. K., & Tariq, S. (2024). The Inverted Nadarajah-Haghghi Power Series Distributions. *International Journal of Applied and Computational Mathematics*, 10(1), 11.

Ebere, A. K., Ifeanyi, E. E., Chukwudi, O. J & Ebuka, D. M. (2024). New Generalized Nadarajah-Haghghi distribution: Characterization and Application. *Journal of Modern Applied Statistical Methods*, 23 (1), 1-17. DOI: 10.56801/Jmasm.V23.i1.11

Hiba, Z., & Hagar, M. (2023). A modified weighted Rayleigh distribution. *Journal of probability and statistical sciences*, 1 (21), 60-83. DOI:10.37119/jpss2023.v21i1.637

Nadarajah, S., & Haghghi, F. (2011). Nadarajah-Haghghi distribution. *A journal of heoretical and Applied statistics*, 6 (45), 543-558.

Osi, A. A, Sabo, S. A., & Musa, I. Z., (2024). Inverted Dagum distribution: properties and application to lifetime dataset Reliability: Theory & Applications 19(3(79)), 595-604.



- Osi, A., Raghav, Y. S., Sabo, S. A., & Musa, I. Z. (2025). The new transformed Sine G family of distribution with inferences and application. *Brazilian Journal of Biometrics*, 43(3). DOI: <https://doi.org/10.28951/bjb.v43i3.755>
- Pena-Ramirez, F., R.R., G., Canterle, D., & Cordeiro, G. (2020). The logistic Nadarajah-Haghighi distribution. *Reliability engineering and system safety*, 204:107176.
- Sebastian, N., Joseph, J., Muhsina, C. S., & Sandra, I. S. (2024). Marshall-Olkin Exponentiated Nadarajah Haghighi Distribution and its Applications. *Reliability: Theory & Applications*, 19(4 (80)), 105-119.
- Singh, B., Alam, I., Rather, A. A., & Alam, M. (2023). Linear combination of order statistics of exponentiated Nadarajah-Haghighi distribution and their applications. *Lobachevskii Journal of Mathematics*, 44(11), 4839-4848. DOI: <https://doi.org/10.1134/S1995080223110318>
- Usman, A., Ishaq, A. I., Suleiman, A. A., Othman, M., Daud, H., & Aliyu, Y. (2023). Univariate and bivariate log-topp-leone distribution using censored and uncensored datasets . In *Computer Sciences & Mathematics Forum*. 7, pp. 32-38. MDPI. DOI:<https://doi.org/10.3390/IOCMA2023-14421>
- Usman, U., & Aliyu, Y. (2022). Bivariate Nadarajah-Haghighi distribution derived from copula functions: Bayesian estimation and applications. *Benin Journal of Statistics*, 5, 45-57.

APPENDIX

Table A1: Data on Kidney Infection

PATIENT	T ₁	T ₂	SEX	AGE	DISEASE TYPES
1	8	16	1	28	3
2	23	13	0	48	0
3	22	28	1	32	3
4	447	318	0	31.5	3
5	30	12	1	10	3
6	24	245	0	16.5	3
7	7	9	1	51 0	
8	511	30	0	55.5	0
9	53	196	0	69	1
10	15	154	1	51.5	0
11	7	333	0	44	1
12	141	8	0	34	3
13	96	38	0	35	1
14	149	70	0	42	1
15	536	25	0	17	3
16	17	4	1	60	1
17	185	177	0	60	3
18	292	114	0	43.5	3
19	22	159	0	53	0
20	15	108	0	44	3
21	152	562	1	46.5	2
22	402	24	0	30	3
23	13	66	0	62.5	1
24	39	46	0	42.5	1
25	12	40	1	43	1
26	113	201	0	57.5	1
27	132	156	0	10	0
28	34	30	0	52	1
29	2	25	1	53	0
30	130	26	0	54	0
31	27	58	0	56	1
32	5	43	0	50.5	1
33	152	30	0	57	2
34	190	5	0	44.5	0
35	119	8	0	22	3



IJSGS FUGUSAU VOLXXX **WEBSITE: <https://fugus-ijsgs.com.ng>**

36	54	16	0	42	3
37	6	78	0	52	2
38	63	8	1	60	2

Table A2: Undergraduate Students Scores Data

S/N	1	2	3	4	5	6	7	8	9	10	11	12	13	14
T_1	62	33	50	50	67	57	56	50	50	50	58	67	53	30
T_2	54	50	60	61	58	68	58	62	39	25	89	32	33	63