

# On the Graph Theoretic Properties of the Inverse-Order Sum Graph of finite groups: a new Graph Constructed from Inverse and Order Sum Graphs of $Z_n$

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Received on: January, 2025

Revised and Accepted on: February, 2025

Published on: March, 2025

## ABSTRACT

This paper introduces the inverse-order sum graph of the group  $Z_n$ ,  $\Gamma_{IOSG}(Z_n)$ , a new graph structure that combines the properties of the inverse graph and the order sum graph. We define the inverse-order sum graph of  $Z_n$  as a simple graph whose vertices are the elements of  $Z_n$ , and two distinct vertices are adjacent if either the sum of their orders is greater than or equal to the order of  $Z_n$  or they are inverses of each other. By combining these two properties, we create a graph that provides a visual representation of the intricate relationships between elements of  $Z_n$ . We investigate the graph theoretic properties of the inverse-order sum graph of the cyclic group  $Z_n$ , obtaining results on connectivity, completeness, regularity, vertex degree, and graph size. Findings indicated that for  $n$  being an odd prime,  $\Gamma_{IOSG}(Z_n)$  is connected, not complete, and not regular. Specifically, the vertex degree and the graph size were obtained.

**Keywords:** graph theory, connectivity, completeness, regularity, vertex degree

## 1.0 INTRODUCTION

### 1.1 Introduce the Problem

There are several abstract algebra concepts which have been studied and facilitated by the construction of graphs. On the other hand, many properties of abstract algebraic structures have been characterized with the help of graph theory. Graphical representation of algebraic structures have been studied extensively and has become a research topic which is energizing over the recent couple of decades. These studies have raised several intriguing outcomes and a lot of research questions leading to more research areas and increasing interest by researchers (Ajay, Lavanya, Peter, & Tamizh, Recent developments on the power graph of finite groups – a survey., 2021). The study of graphs associated with groups has been a fertile area of research in algebraic graph theory. In particular, the inverse graph (Rian, Salman, & Pritta, 2023) and the order sum graph (Javeria & Sudev, 2023) have been extensively studied in the context of finite groups. However, the relationship between these two graphs has not been fully explored.

Previous studies have investigated the properties of some graphs such as Non-commuting graph (Abdollahi, Akbari, & Maimani, 2006), Prime Coprime graph (Adhikari & Banerjee, 2022), Order Graph (Al-Hasanat, Al-Hasanat, & Ma'an, 2019), Subgroup Graph (Ejima, Garba, & Aremu, 2021), non-inverse graph (Javeria & Sudev, 2022), inverse graph (Rian, Salman, & Pritta, 2023) and the order sum graph (Mahera, Akram, & Nabeel, 2023). All these graph were studied separately. The

combination of two properties in a single graph to form a new graph has been studied by authors such as (Magami, Ibrahim, Ashafa, & Gana, 2024) where identity graph and commuting graph were combined, (Garba, Magami, & Ejima, 2021) where identity graph and power graph were combined etc. However the combination of the inverse graph and order sum graph has not been explored.

Motivated by the work of Garba, Magami, and Ejima (2021), where they investigate some algebraic properties and combinatorial structures of the identity-power graph of group. In Garba, Magami, and Ejima (2021) created the identity-power graph using two different graphs of group, namely, the identity graph of groups and the power graphs of groups. Another research by Magami, Ibrahim, Ashafa, and Gana (2024) where identity graph and commuting graph were combined investigates some algebraic properties and combinatorial structures of the identity-commuting graph of multigroup which include: vertex degree, size, connectedness, completeness among others. It was shown that the graph is connected but not complete and condition for completeness was obtained.

Despite the extensive research on graph structures associated with groups, such as the identity-power graph (Garba, Magami, & Ejima, 2021) and the identity-commuting graph (Magami, Ibrahim, Ashafa, & Gana, 2024), there is a notable absence of a graph structure that combines the properties of the inverse graph and the order sum graph, highlighting a significant gap in the literature. This study builds on previous work by introducing a new graph structure that combines the properties of the inverse graph and the order sum graph. This study aims to fill this gap by introducing the inverse-order sum graph of a finite

group  $G$ , denoted by  $\Gamma_{IOSG}(G)$ , which combines the properties of the inverse graph and the order sum graph. The inverse-order sum graph is important because it provides a visual representation of the intricate relationships between elements of a group. By studying the properties of this graph, we can gain insights into the structure and behavior of the underlying group. This study focuses on the case where  $G$  is the cyclic group  $Z_n$ , and investigates the graph theoretic properties of  $\Gamma_{IOSG}(Z_n)$ .

**2.0 MATERIAL AND METHOD**

**2.1 Material**

This section deals with some graph theoretic terminologies together with some of the results obtained previously and applied to achieve our the main results of this study. Moreover, the terminologies assist in the understanding of our paper. The graphs considered in this work are simple graphs; they are undirected graphs that do not have multiple edges or loops.

A graph  $\Gamma$  is formed by finite non-empty pair of sets  $V$  and  $E$ .  $V$  is the set of *vertices* which can be viewed as points while  $E$  is the set of *edges* which are line connecting these points. The number of elements in  $V$  is called the *order* of  $\Gamma$  and the total number of edges in  $\Gamma$  is called the *size* of  $\Gamma$ .  $d(v)$  represents the *degree* of a vertex  $v \in V$  in the graph  $\Gamma$  which is the number of edges connected to it (Garba, Magami, & Ejima, 2021).  $\delta(\Gamma)$  represents the *minimum* vertex degree in a graph  $\Gamma$  while  $\Delta(\Gamma)$  represents the *maximum* vertex degree in the graph  $\Gamma$ . Let  $S \subseteq V$ , then  $\Gamma[S]$  is the induced subgraph of  $\Gamma$  and it is the graph with vertices set  $S$  and edges with both ends in  $S$ . A subset of  $V$  in which no two vertices are connected is called an *independent* set (Ajay, Lavanya, Peter, & Tamizh, 2021).

**Definition 2.1.** (Rian, Salman, & Pritta, 2023). Inverse Graph of a group  $G$  is a simple graph whose vertex set is  $G$  and two distinct vertices,  $u$  and  $v$  in  $G$ , are adjacent if  $u * v \in T_r$  or  $v * u \in T_r$  where  $T_r = \{t \in G \mid t \neq t^{-1}\}$ . The inverse graph of group  $G$  is denoted by  $\Gamma_{IV}(G)$ .

**Example 2.1.** Consider the group  $A_4 = \{(1), (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}$ , which is the alternating group whose all members are even permutations of four objects then  $\Gamma_{IV}(A_4)$  is given in the following figure:

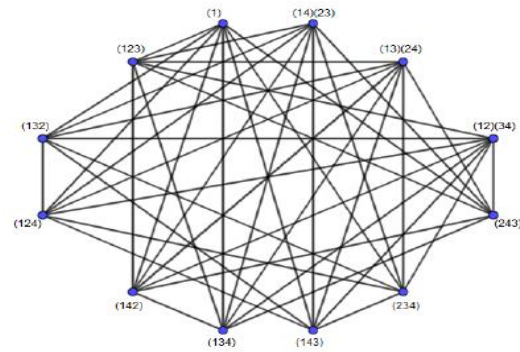


Figure 1: Inverse Graph of  $A_4$

**Definition 2.2.** (Javeria & Sudev, 2023). Order sum Graph of a group  $G$  is a simple graph whose vertices are the elements of  $G$ , and two distinct vertices are adjacent if either the sum of their orders is greater than or equal to the order of  $G$ . The order sum graph of group  $G$  is denoted by  $\Gamma_{OS}(G)$ .

**Example 2.2.** Consider the group  $Z_6 = \{0,1,2,3,4,5\}$ , then  $\Gamma_{OS}(Z_6)$  is given in the following figure:

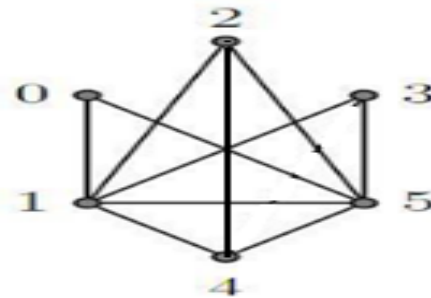


Figure 2: Order Sum Graph of  $Z_6$

**Definition 2.3.** The inverse-order sum graph of  $Z_n$ , denoted by  $\Gamma_{IOSG}(Z_n)$ , is a graph whose vertices are the elements of  $Z_n$ , and two distinct vertices are adjacent if either the sum of their orders is greater than or equal to the order of  $Z_n$  or they are inverses of each other.

**Example 2.3.**  $\Gamma_{IOSG}(Z_6)$  is given in the following figure:

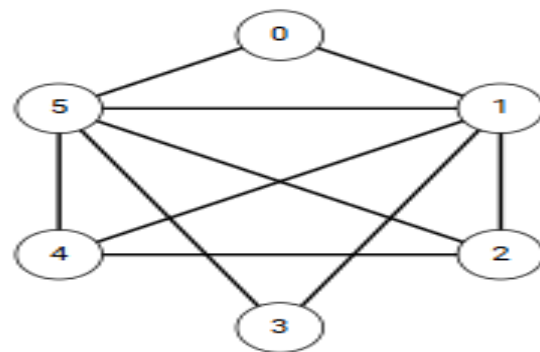


Figure 3: Inverse - Order Sum Graph of  $Z_6$

**Definition 2.4.** A graph  $\Gamma$  is said to be complete, if every pair of its distinct vertices are adjacent. A complete graph on  $n$  vertices is denoted by  $K_n$  (Ajay, Lavanya, Peter, & Tamizh, 2021).

**Definition 2.5.** A graph  $\Gamma$  is said to be connected if any two distinct vertices of  $\Gamma$  are joined by a path.  $\Gamma$  is said to be disconnected if  $\Gamma$  is not connected (Ajay, Lavanya, Peter, & Tamizh, 2021).

**Definition 2.6.** (Self Inverse Elements). Let  $(X, *)$  be a finite Abelian group, an element  $x \in X$  is called *self inverse* element if  $x^{-1} = x$ . The set of all self inverse elements in  $X$  is denoted by  $S_{inv}$  (Sowaity, Sharada, & Naji, 2020).

**Definition 2.7.** (Mutual Inverse Elements). Let  $(X, *)$  be a finite Abelian group, an element  $y \in X$  is called *mutual inverse* element if  $\exists y' \in X$  with  $y \neq y'$  such that  $yy' = y'y$ . The set of all mutual inverse elements in  $X$  is denoted by  $M_{inv}$  (Sowaity *et al.*, 2020).

**Notation 2.8.** Let  $G$  be a multigroup over the finite Abelian group  $X$ , then we have the following notations:

1.  $x \sim y$  denotes that the 2 vertices  $x$  and  $y$  are adjacent, and given a set of vertices  $V$ , then we write  $x \sim V$  if a vertex  $v$  is adjacent to every vertex in a set  $V$ .
2.  $V(\Gamma)$  and  $E(\Gamma)$  denotes the set of vertices and set of edges of the graph  $\Gamma$  respectively.
3.  $|S|$  represents the number of elements in the set  $S$ .

## 2.2 Method

### Lemma 1: Self-Inverse Elements in Abelian Groups (Sowaity, Sharada, & Naji, 2020)

In an Abelian group  $G$ , the number of self-inverse elements,  $|S_{inv}|$  is determined by the parity of the group's order  $n$ . Specifically:

$$|S_{inv}| = \begin{cases} 2, & \text{if } n = \text{even} \\ 1, & \text{if } n = \text{odd} \end{cases}$$

### Lemma 2: Self-Inverse Elements in finite Groups $Z_n$

Consider the finite group  $Z_n = \{0, 1, 2, \dots, n-1\}$ . The number of self-inverse elements  $|S_{inv}|$  in  $Z_n$  depends on the parity of  $n$ :

$$|S_{inv}| = \begin{cases} \left\{0, \frac{n}{2}\right\}, & \text{if } n = \text{even} \\ 0, & \text{if } n = \text{odd} \end{cases}$$

#### Proof

For  $Z_n$  under integer modulo  $n$ , we have the following cases:

Case 1: If  $n$  is even, then for  $x \in Z_n$ ,  $x + x^{-1} = n$ , implying  $x^{-1} = n - x$ . Self-inverse elements satisfy  $x + x \equiv 0 \pmod{n}$ , leading to  $2x \equiv n \pmod{n}$ . This yields two solutions:  $x = 0$  and  $x = \frac{n}{2}$ .

Case 2: If  $n$  is odd, Lemma 1 guarantees only one self-inverse element,  $x = 0$ , since  $0^{-1} = 0$ .

### Lemma 3: Partitioning $Z_n$ for $n = 2p$ ( $p > 2$ and $p$ prime)

The elements of  $Z_n$ , where  $n = 2p$  ( $p > 2$ ), can be partitioned into four disjoint subsets:

1.  $P_1 = \{0\}$  with each element having order 1 with  $|P_1| = 1$ .
2.  $P_2 = \left\{\frac{n}{2}\right\}$  with each element having order 2 with  $|P_2| = 1$ .
3.  $P_3 = \{\text{even numbers}\}$  with each element having order  $p$  with  $|P_3| = \frac{n-2}{2}$ .
4.  $P_4 = \left\{\text{odd numbers} \setminus \left\{\frac{n}{2}\right\}\right\}$  with each element having order  $2p$  with  $|P_4| = \frac{n-2}{2}$ .

#### Proof:

To prove this lemma, we need to show that the four subsets  $P_1, P_2, P_3$ , and  $P_4$  are disjoint and that every element of  $Z_n$  belongs to one of these subsets.

#### Disjointness of the subsets

The subsets  $P_1, P_2, P_3$ , and  $P_4$  are disjoint. Specifically,  $P_1$  and  $P_2$  are disjoint since  $0 \neq \frac{n}{2}$ . Additionally,  $P_1$  and  $P_3$  are disjoint because 0 is not an even number greater than 0. Furthermore,  $P_1$  and  $P_4$  are disjoint since 0 is not an odd number. The disjointness continues with  $P_2$  and  $P_3$ , as  $\frac{n}{2} = p$  is not an even number, where  $p > 2$  and  $p$  is prime. Also,  $P_2$  and  $P_4$  are disjoint since  $\frac{n}{2} \notin P_4$  by definition of  $P_4$ . Lastly,  $P_3$  and  $P_4$  are disjoint by definition.

#### Order of elements in each subset

The order of elements in each subset is as follows. The subset  $P_1$ , containing only the identity element 0, has an order of 1. In  $P_2$ , the element  $\frac{n}{2}$  has an order of 2, as  $\frac{n}{2} + \frac{n}{2} = n \equiv 0 \pmod{n}$ . For elements in  $P_3$ , which are even numbers, the order is  $p$ . This is because for any  $x$  in  $P_3$ ,  $x$  can be expressed as  $2k$  for some integer  $k$ , and  $x^p = 2kp \equiv 0 \pmod{n}$ . Moreover, if  $x \cdot m \equiv 0 \pmod{n}$  for some  $m < p$ , it leads to a contradiction, implying that the order of  $x$  is indeed  $p$ . Lastly, for elements in  $P_4$ , which are odd numbers excluding  $\frac{n}{2}$ , the order is  $2p$ . This is because for any  $x$  in  $P_4$ ,  $x$  can be expressed as  $2k + 1$  for some integer  $k$ , and  $x \cdot 2p \equiv 0 \pmod{n}$ . Similar to the previous case, if  $x \cdot m \equiv 0 \pmod{n}$  for some  $m < 2p$ , it leads to a contradiction, implying that the order of  $x$  is indeed  $2p$ .



**Cardinality of each subset**

The cardinalities of the subsets  $P_1, P_2, P_3$ , and  $P_4$  are determined as follows. The cardinality of  $P_1$  is 1, as it contains only the single element 0. Similarly, the cardinality of  $P_2$  is also 1, since it contains only the element  $\frac{n}{2}$ . The cardinality of  $P_3$  is  $\frac{n-2}{2}$ , as it comprises all even numbers in  $Z_n$  excluding 0 and  $\frac{n}{2}$ , and there are  $\frac{n-2}{2}$  even numbers between 2 and  $n-2$ , inclusive. Likewise, the cardinality of  $P_4$  is also  $\frac{n-2}{2}$ , as it consists of all odd numbers except  $\frac{n}{2}$ , and there are  $\frac{n-2}{2}$  odd numbers between 1 and  $n-1$ , inclusive, excluding  $\frac{n}{2}$ .

Hence the result.

The above stated lemmas are crucial in establishing the graph theoretic properties of  $\Gamma_{IOSG}(Z_n)$ .

**3.0 RESULTS AND DISCUSSION**

Since  $v = 0$  then we have the following table:

$v$ is adjacent to:	$P_4$	$o(0) + o(x) = 1 + 2p \geq 2p$ for $x \in P_4$
$v$ is not adjacent to:	$P_1$	$v$ is self inverse and there is no loop
	$P_2$	$o(0) + o(y) = 1 + 2 < 2p$ for $y \in P_2$ (since $p > 2$ )
	$P_3$	$o(0) + o(y) = 1 + p < 2p$ for $y \in P_3$ (since $p > 2$ )

Hence,  $\deg(v) = |P_4| = \frac{n-2}{2}$ .

**Case 2:  $v \in P_2$  then  $v = \frac{n}{2}$  and  $o(v) = 2$ .**

By definition of  $\Gamma_{IOSG}(Z_n)$ , every vertex is connected to:

1. Its inverse vertex.
2. Any vertex whose order, when added to its own order, results in a value greater than or equal to the order of the group.

Since  $v = \frac{n}{2}$  then we have the following table:

$v$ is adjacent to:	$P_4$	$o(v) + o(x) = 2 + 2p \geq 2p$ for $x \in P_4$
$v$ is not adjacent to:	$P_1$	$o(v) + o(y) = 2 + 1 = 3 < 2p$ for $y \in P_1$ (since $p > 2$ )
	$P_2$	$v$ is self inverse and there is no loop
	$P_3$	$o(v) + o(y) = 2 + p < 2p$ for $y \in P_3$ (since $p > 2$ )

Hence,  $\deg(v) = |P_4| = \frac{n-2}{2}$ .

**Case 3:  $v \in P_3$  then  $v = \text{even number}$  and  $o(v) = p$ .**

By definition of  $\Gamma_{IOSG}(Z_n)$ , every vertex is connected to:

1. Its inverse vertex.
2. Any vertex whose order, when added to its own order, results in a value greater than or equal to the order of the group.

Since  $v = \frac{n}{2}$  then we have the following table:

$v$ is adjacent to:	$P_3 \setminus \{v\}$	$o(v) + o(x) = p + p \geq 2p$ for $x \in P_3$
	$P_4$	$o(v) + o(y) = p + 2p < 2p$ for $y \in P_4$ (since $p > 2$ )
$v$ is not adjacent to:	$P_1$	$o(v) + o(z) = 2 + 1 = 3 < 2p$ for $z \in P_1$ (since $p > 2$ )
	$P_2$	$v^{-1} \notin P_2$ and $o(v) + o(y) = p + 2 < 2p$ for $y \in P_2$

Hence,  $\deg(v) = |P_3 \setminus \{v\}| + |P_4| = \frac{n-2}{2} - 1 + \frac{n-2}{2} = n - 3$ .

**Case 4:  $v \in P_4$  then  $v \in \left\{ \text{odd number} \setminus \left\{ \frac{n}{2} \right\} \right\}$  and  $o(v) = 2p$ .**

By definition of  $\Gamma_{IOSG}(Z_n)$ , every vertex is connected to:

**Theorem 3.1: Degree of Vertices in Inverse-Order Sum Graphs**

Let  $\Gamma_{IOSG}(Z_n)$  be the graph of the inverse-order sum graph, where  $n = 2p$ ,  $p$  is a prime number greater than 2, and  $Z_n$  is partitioned as specified in Lemma 3. Then, for any vertex  $v$  in  $\Gamma_{IOSG}(Z_n)$ , the degree of  $v$  is:

$$\deg(v) = \begin{cases} \frac{n-2}{2}, & v \in P_1 \cup P_2 \\ n - 3, & v \in P_3 \\ n - 1, & v \in P_4 \end{cases}$$

**Proof:**

Suppose  $\Gamma_{IOSG}(Z_n)$  is the inverse-order sum graph, where  $n = 2p$ ,  $p$  is a prime number greater than 2, and  $Z_n$  is partitioned as specified in Lemma 3. Let  $v$  be an arbitrary vertex in  $\Gamma_{IOSG}(Z_n)$ , then we consider the following cases:

**Case 1:  $v \in P_1$  then  $v = 0$  and  $o(v) = 1$ .**

By definition of  $\Gamma_{IOSG}(Z_n)$ , every vertex is connected to:

1. Its inverse vertex.
2. Any vertex whose order, when added to its own order, results in a value greater than or equal to the order of the group.

1. Its inverse vertex.
2. Any vertex whose order, when added to its own order, results in a value greater than or equal to the order of the group.

Since  $v = \{ \text{odd numbers} \setminus \{ \frac{n}{2} \} \}$ , then we have the following table:

$v$ is adjacent to:	$P_1$ $P_2$ $P_3$ $P_4 \setminus \{v\}$	$o(v) + o(x) = 2p + 1 \geq 2p$ for $x \in P_1$ (since $p > 2$ ) $o(v) + o(y) = 2p + 2 \geq 2p$ for $y \in P_2$ (since $p > 2$ ) $o(v) + o(z) = 2p + p \geq 2p$ for $z \in P_3$ (since $p > 2$ ) $o(v) + o(r) = 2p + 2p \geq 2p$ for $r \in P_4$ (since $p > 2$ )
$v$ is not adjacent to:	<i>Nil</i>	

Hence,  $\text{deg}(v) = |Z_n| - 1 = n - 1$ .

$$\text{Hence the result } \text{deg}(v) = \begin{cases} \frac{n-2}{2}, v \in P_1 \cup P_2 \\ n-3, v \in P_3 \\ n-1, v \in P_4 \end{cases}$$

**Example:** Consider  $\Gamma_{IOSG}(Z_{10})$ , the vertices are  $Z_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$  with partitions  $P_1 = \{0\}$ ;  $P_2 = \{5\}$ ;  $P_3 = \{2, 4, 6, 8\}$ ;  $P_4 = \{1, 3, 7, 9\}$ . The Degree of Vertices (by Theorem 3.1):

- $\text{deg}(0) = \frac{n-2}{2} = 4$  (adjacent to  $P_4$ )
- $\text{deg}(5) = \frac{n-2}{2} = 4$  (adjacent to  $P_4$ )
- $\text{deg}(2) = n - 3 = 7$  (adjacent to  $P_3 \setminus \{2\}$ ,  $P_4$ , and its inverse 8)
- $\text{deg}(3) = n - 1 = 9$  (adjacent to all vertices except itself)

Please refer to the Figure 4.1 for a visual representation of the graph.

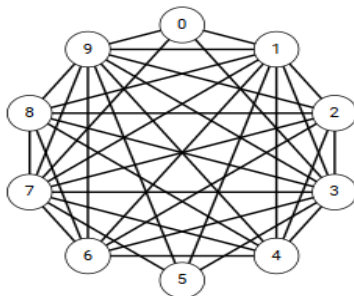


Figure 3.1: Inverse – Order Sum Graph of  $Z_{10}$

The degree of vertices in the inverse-order sum graph plays a crucial role in understanding the graph's structure. Theorem 3.1 provides a formula for calculating the degree of vertices, which depends on the partition to which the vertex belongs. This result has implications for the graph's connectivity and regularity, which are explored in the following theorems.

**Theorem 3.2: Graph Size of Inverse-Order Sum Graphs**

Let  $\Gamma_{IOSG}(Z_n)$  be the inverse-order sum graph of  $Z_n$ , where  $n = 2p$ ,  $p$  is a prime number greater than 2. Then, the graph size is given by:

$$|E(\Gamma_{IOSG}(Z_n))| = \frac{n^2 - 3n + 2}{2}$$

**Proof:**

By definition of  $\Gamma_{IOSG}(Z_n)$ , the edges of  $\Gamma_{IOSG}(Z_n)$  can be classified into three classes:

$E_1$ : Edges within the complete subgraph of  $P_4$  ( $\forall x, y \in P_4, o(x) + o(y) = 2p + 2p \geq 2p$ )

$E_2$ : Edges within the complete subgraph of  $P_3$  ( $\forall x, y \in P_3, o(x) + o(y) = p + p \geq 2p$ )

$E_3$ : Edges between  $P_4$  and other partitions ( $P_1, P_2, P_3$ )

Using the Handshaking Lemma, we calculate:

$$E_1 = \frac{|P_4|(|P_4|-1)}{2}; E_2 = \frac{|P_3|(|P_3|-1)}{2}; E_3 = |P_4|(|P_1| + |P_2| + |P_3|)$$

Where,  $|P_4| = |P_3| = \frac{n-2}{2}$ ,  $|P_2| = \frac{n}{2}$  &  $|P_1| = 1$

Combining these, we get:

$$|E(\Gamma_{IOSG}(Z_n))| = E_1 + E_2 + E_3 = \frac{n^2 - 3n + 2}{2}$$

**Example.** Consider  $\Gamma_{IOSG}(Z_6)$ , the vertices are  $Z_6 = \{0, 1, 2, 3, 4, 5\}$  with partitions:  $P_1 = \{0\}$ ,  $P_2 = \{3\}$ ,  $P_3 = \{2, 4\}$ , and  $P_4 = \{1, 5\}$ .

Calculating the edges:  $E_1 = \frac{|P_4|(|P_4|-1)}{2} = \frac{2(1)}{2} = 1$  ;

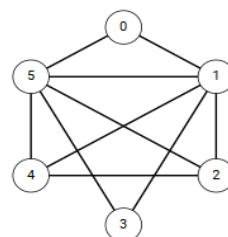
$E_2 = \frac{|P_3|(|P_3|-1)}{2} = \frac{2(1)}{2} = 1$ ;

$E_3 = |P_4|(|P_1| + |P_2| + |P_3|) = 2(1 + 1 + 2) = 8$

Total edges:  $|E(\Gamma_{IOSG}(Z_6))| = E_1 + E_2 + E_3 = 1 + 1 + 8 = 10$

Verifying the formula:  $|E(\Gamma_{IOSG}(Z_6))| = \frac{n^2 - 3n + 2}{2} = 10$

Please refer to the Figure 3.2 for a visual representation



**Figure 3.2: Inverse - Order Sum Graph of  $Z_6$** 

The graph size of the inverse-order sum graph is an important invariant that provides insight into the graph's complexity. Theorem 3.2 shows that the graph size can be calculated using a simple formula, which depends on the order of the cyclic group. This result has implications for the graph's connectivity, which is explored in the next theorem.

**Theorem 3.3: Connectedness of Inverse-Order Sum Graphs**

Let  $n = 2p$ , where  $p$  is an odd prime number greater than 2. Then, the inverse-order sum graph  $\Gamma_{IOSG}(Z_n)$  is connected.

**Proof:**

Consider  $\Gamma_{IOSG}(Z_n)$  with  $n = 2p$ ,  $p$  being an odd prime number greater than 2. The graph is partitioned as specified by Lemma 3. By definition of  $\Gamma_{IOSG}(Z_n)$ , every vertex in partition  $P_4$  is adjacent to all other vertices in the graph. This is because the order of each vertex in  $P_4$  is  $2p$ , and when added to the order of any other vertex, the sum exceeds  $2p$ , satisfying the connection criterion. So, for any two distinct vertices  $x, y \in V(\Gamma_{IOSG}(Z_n))$ , we examine two cases:

**Case 1:**  $x, y \in P_4$ , then  $o(x) = o(y) = 2p$ .  $x \sim y$  since  $o(x) + o(z) = 2p + 2p \geq 2p$ . Therefore a path exists between  $x$  and  $y$ :  $x \sim y$ .

**Case 2:**  $x, y \notin P_4$  then  $o(x), o(y) \in \{1, 2, p\}$ . There exists a path connects  $x$  and  $y$  through  $P_4$  and this can be seen by choosing  $z \in P_4$  with  $o(z) = 2p$ . Then:  
 $x \sim z$  ( $o(x) + o(z) \geq 2p$ ) and  $y \sim z$  ( $o(y) + o(z) \geq 2p$ ). Therefore  $x \sim z \sim y$ .  
 Since  $x$  and  $y$  are arbitrary,  $\Gamma_{IOSG}(Z_n)$  is connected.

**Example:** Consider  $\Gamma_{IOSG}(Z_6)$  with  $Z_6 = \{0, 1, 2, 3, 4, 5\}$ . The graph is connected as seen in the Figure 3.2.

The connectedness of the inverse-order sum graph is a fundamental property that ensures that the graph is a single, cohesive unit. Theorem 3.3 shows that the graph is indeed connected, regardless of the order ( $n = 2p$ ) of the cyclic group. This result has implications for the graph's completeness and regularity, which is explored in the next two theorems.

**Theorem 3.4: Incomplete Nature of Inverse-Order Sum Graphs**

Let  $n = 2p$ , where  $p$  is an odd prime number greater than 2. Then, the inverse-order sum graph  $\Gamma_{IOSG}(Z_n)$  is not a complete graph.

**Proof**

To prove this, we show that a disconnected pair of vertices always exists in  $\Gamma_{IOSG}(Z_n)$  if  $n = 2p$ , where  $p$  is an odd prime number greater than 2. Consider  $x \in P_1$  and  $y \in P_2$ , then  $o(x) = 1$  (order of  $x$ ) and  $o(y) = 2$  (order of  $y$ ). Since  $o(x) \neq o(y)$  then  $x$  and  $y$  are not inverse of each other. Moreover since  $(x) + o(y) = 3 < 2p$  ( $p > 2$ ), then by definition of  $\Gamma_{IOSG}(Z_n)$ ,  $x$  and  $y$  are not adjacent. This disconnected pair of vertices implies that  $\Gamma_{IOSG}(Z_n)$  cannot be a complete graph.

**Example:** Consider  $\Gamma_{IOSG}(Z_{10})$  with  $Z_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ . The partitions are:  $P_1 = \{0\}$ ;  $P_2 = \{5\}$ ;  $P_3 = \{2, 4, 6, 8\}$ ;  $P_4 = \{1, 3, 7, 9\}$

**Check;** Choose  $x = 0 \in P_1$  and  $y = 5 \in P_2$ , then  $o(0) = 1$  and  $o(5) = 2$ . However,  $0 \neq 5^{-1}$  and  $o(0) + o(5) = 1 + 2 = 3 < 2(5) = 10$ . Therefore, by definition of  $\Gamma_{IOSG}(Z_{10})$ ,  $0 \not\sim 5$ . This example demonstrates that  $\Gamma_{IOSG}(Z_{10})$  is not a complete graph.

The completeness of a graph is a measure of how interconnected the graph is. Theorem 3.4 shows that if  $n = 2p$ , where  $p$  is a prime number greater than 2 then the inverse-order sum graph of  $Z_n$  is never complete. This result highlights the graph's complex structure and provides insight into its properties.

**Theorem 3.5: Regularity of Inverse-Order Sum Graphs**

Let  $n = 2p$ , where  $p$  is an odd prime number greater than 2. Then, the inverse-order sum graph  $\Gamma_{IOSG}(Z_n)$  is not a regular graph.

**Proof:**

Suppose  $n = 2p$ , where  $p$  is an odd prime number greater than 2 and  $\Gamma_{IOSG}(Z_n)$  is the inverse-order sum graph of  $Z_n$ . Let the vertices of  $\Gamma_{IOSG}(Z_n)$  be partitioned as specified by Lemma 3, then to prove the regularity, we need to show that every vertex has the same degree. We consider the following cases;

**Case 1:** Vertices in  $P_1 \cup P_2$   

$$\deg(v) = \frac{n-2}{2}, v \in P_1 \cup P_2$$

**Case 2:** Vertices in  $P_3$   

$$\deg(v) = n-3, v \in P_3$$

**Case 3:** Vertices in  $P_4$   

$$\deg(v) = n-1, v \in P_4$$

Although the degrees vary across different partitions, but within each partition, the degree is the same. Therefore, the  $\Gamma_{IOSG}(Z_n)$  graph is not regular, as vertices in different partitions have different degrees for instance  $n-3 \neq n-1$  because  $n = 2p$  with  $p > 2$ . However, within each partition, the graph is regular. Hence the result.

This result highlights the graph's complex structure and provides insight into its properties. The regularity of the graph within each partition is an interesting property that warrants further investigation.

#### 4.0 CONCLUSION

This study investigated the properties of the inverse-order sum graph of the cyclic group  $Z_n$ , where  $n = 2p$  and  $p$  is an odd prime number greater than 2. The results of this study provide new insights into the structure of these graphs, which have not been previously studied.

The main purpose of this study was to determine the degree of vertices, graph size, connectedness, completeness, and regularity of the inverse-order sum graph of  $Z_n$ . The results show that the degree of vertices depends on the partition to which the vertex belongs, the graph size can be calculated using a simple formula, the graph is connected but not complete, and the graph is not regular.

The results of this study have significant implications for our understanding of the structure of inverse-order sum graphs. The complex structure of these graphs makes them ideal for use in cryptographic protocols.

This study opens up several avenues for future research, including investigating the graph theoretic properties of the inverse-order sum graph for other types of groups, such as abelian or non-abelian groups. Additionally, examining the relationship between the inverse-order sum graph and other graph structures, like the Cayley graph or the Schreier graph, presents another potential direction. Furthermore, applying the results of this study to real-world problems, such as network analysis or cryptography, offers opportunities for practical applications. Future research should investigate the structure of inverse-order sum graphs for other values of  $n$ .

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