

Modified Kumaraswamy Negative Exponential Distribution

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ABSTRACT

In this research a four parameter continuous probability distribution model called Kumaraswamy negative exponential distribution (KNED) was proposed as an enhancement of the negative exponential distribution. The probability density function and cumulative distribution function of the modified distribution were derived. Additionally, the validity of this distribution was demonstrated. Also, various properties of the modified distribution for instance survival and hazard functions, moments, moment generating functions, and order statistics were obtained. Parameters for the modified distribution were estimated using maximum likelihood estimation method. Furthermore, the proposed model was compared to other variations of exponential distribution using some analytical measures of goodness of fit on information criteria and the results obtained shows that our proposed model gives better fit, thus proving greater flexibility in modelling real-world phenomenon.

Keywords: Kumaraswamy-G family, negative exponential distribution, Kumaraswamy negative exponential distribution, moment, survival function.

1.0 INTRODUCTION

Statistical probability distribution theory is a field of statistics that focuses on examining, defining, and analysing probability distributions. A probability distribution explains the way in which the potential values of a random variable are spread out or distributed. It offers crucial insights into the probabilities of various results in a random process or experiment. There are many types of statistical distributions, each with its own characteristics and parameters.

These include the: normal (Gaussian) distribution, binomial distribution, Poisson distribution, exponential distribution, etc. They play a crucial role in modelling and understanding real-world data and are widely applied in diverse fields, for example, economics, engineering, biology, finance, social sciences, etcetera. They also aid in representing characteristics of data, including asymmetry, bimodality, or multimodality commonly seen in various fields like engineering, medicine, and finance.

Sun (2011) stated that the flexibility of a distribution can be enhanced through generalization, which involves utilizing available generalized families of distributions. When a distribution is generalized, additional shape parameters are incorporated from the family of distributions employed. These extra shape parameters serve to adjust the tail weight of the resulting compound distribution, thereby introducing skewness. Flexibility can also be increased by modifying existing distributions. For example, combining two or more standard distributions through convolution, quotient, or product of independent random variables. Furthermore, some distributions are

found to be functions of one or more other distributions, as exemplified by the composition of the student t-distribution.

Usman *et al.* (2019), stated that in the field of statistical modelling, many real-life datasets with skewed distributions do not conform to commonly use standard probability distribution models. For instance, an exponential distribution with a constant failure rate and predefined shape is not suitable for modelling a dataset with a non-monotonic hazard rate function. Similarly, the Rayleigh distribution is inappropriate for representing symmetric datasets.

George (2014) expressed that the negative exponential distribution is used routinely as a survival distribution, describing the lifetime of a piece of equipment etc., put in service at what may be termed zero. He added that when considering the likelihood of a piece of equipment lasting for an additional t units of time after already surviving s units, one can utilize the negative exponential distribution to obtain the answer. However, we were inspired by prior research (Noman, 2020) employing a four-parameter distribution. This four-parameter model outperformed its sub-models with fewer parameters. Additionally, other scholars have demonstrated that distributions with more parameters possess greater flexibility in modeling reliability and survival data compared to their sub-models with fewer parameters.

Cordeiro and Castro (2011) defined the cdf $F(x)$ and the pdf $f(x)$ of the Kumaraswamy generalized (Kw-G) to extend several widely-known distributions such as the normal, weibull, gamma and Gumbel distributions, in

which they described that major benefit of the distribution is its ability of fitting skewed data that cannot be properly fitted by existing distributions.

Umar *et al.* (2020) proposed an alternative approach to defining the probability density function of Generalized Weibull-exponential distributions, referred to as the Weibull exponentiated exponential distribution. The distribution encompasses the exponential, Weibull, and exponentiated exponential distributions as special cases. The study also provides a comprehensive mathematical analysis of the distribution, including the derivation of the quantile function, mode, characteristic function, moment generating function, and other important mathematical properties. Saboor *et al.* (2021), Olalekan *et al.* (2021), and Ayed *et al.* (2022) also work on the extension of Kumaraswamy distribution.

Usman *et al.* (2019) modified a continuous probability distribution, referred to as the Burr X-Topp Leone distribution, which is presented as an enhancement of the Topp-Leone distribution. However, the density function and cumulative function of the constructed distribution were derived.

Ahmed and Morad (2020) proposed a new family of continuous distributions with three extra shape parameters called the odd Dagum-G family. Matthew *et al.* (2020) proposed generalized Dagum distribution using the quantile function of Lomax distribution. The authors called a member of T-Dagum class of distribution as exponentiated-exponential-Dagum {Lomax} (EEDL) distribution.

Zakari and Usman (2020) developed a continuous probability distribution called Beta-Topp Leone distribution as an enhancement of Topp-Leone distribution. Some of the properties of the enhanced distribution such as; survival and hazard functions, moment, and probability weighted moment and order statistics were derived. Ibrahim *et al.* (2020) stated that when δ less than 1, the distribution has heavy tails, meaning it has a higher probability of extreme values. Conversely, when δ is greater than 1, the tails become lighter?

Tahir *et al.* (2021) proposed a new flexible generalized family (NFGF) for constructing many families of distributions. According to the authors, the importance of the NFGF was that any baseline distribution can be chosen and it does not involve any additional parameters. Musa *et al.* (2021) established some statistical properties and application of Burr X-Topp Leone distribution proposed by (Usman *et al.*, 2019) and the authors used an approach of Burr X family of generator to generate continuous probability distributions. The properties obtained by the authors

include: survival and hazard functions, moments, order statistics, quantile function, probability weighted moment. They used the method of maximum likelihood estimation technique to derive the model parameters. The authors showed the efficiency of BXTLD over some existing methods through application to a real dataset and their result showed that BXTLD performed better in model fitting than some of the life time distributions considered.

Noor and Mundher (2020) proposed the Burr Type X distribution with two parameters by applying the Beta-G, Gamma-G, and Weibull-G distributional families. The authors examined the maximum likelihood estimation of the model parameters. They also evaluated the performance of the parameter estimates through simulation studies under various conditions. Moreover, the authors applied these models to three real-world data sets and concluded that they can be effectively utilized in the analysis of lifetime data.

Manu *et al.* (2023) proposed a three parameter model called Zubair- Kumaraswamy (Z-Kum) distribution. The extension was done using Zubair G-Family of continuous probability distribution to extend the well-known Kumaraswamy distribution and some of the properties of the distributions which includes cdf, pdf, quantile functions, moments, moment generating functions, characteristics functions and order statistics were obtained.

The negative exponential distribution belongs to a two-parameter continuous probability distribution family, which includes scale and location parameters, and it lacks a specific shape parameter that influences the overall form of the distribution.

Thus, to overcome the aforementioned shortcomings, the aim of this study is to develop an alternative four parameters model having additional shape parameter to improve the negative exponential distribution. The novel distribution should be able to effectively model datasets with a non-monotonic hazard rate function, asymmetric, skewed distribution and perform better compared to the baseline, its sub-models and some other related extensions discussed in the literature.

2.0 METHODOLOGY

The baseline distribution used in this research is negative exponential distribution and the generalized family of distribution used is Kumaraswamy-G distribution, in which Kumaraswamy negative exponential distribution was proposed. Some of the properties of the proposed distribution were discussed.

2.1 The CDF and PDF of the baseline distribution

Let x_1, x_2, \dots, x_n be a continuous random variable with probability density function $f(x)$. X is said to have the

Negative exponential distribution if its probability density function (pdf) is defined by:

$$f(x; \varepsilon, \beta) = \frac{1}{\beta} e^{-\frac{(\varepsilon-x)}{\beta}} \quad \text{for } x < \varepsilon,$$

$$-\infty < \varepsilon < \infty, \beta > 0 \quad (1)$$

where ε and β denote the location and scale parameters, respectively.

Then, the corresponding cumulative distribution function is

$$F(x; \varepsilon, \beta) = 1 - e^{-\frac{(\varepsilon-x)}{\beta}}$$

(2)

$$\text{where } \beta > 0$$

2.2 The CDF and PDF of the generalised family distribution

Cordeiro and de Castro (2011) introduced the cumulative distribution function $G(x)$ and probability density function $g(x)$ of the Kumaraswamy generalized distribution.

$$G(x) = 1 - [1 - F(x)^\alpha]^\lambda \quad (3)$$

and

$$g(x) = \alpha \lambda f(x) F(x)^{\alpha-1} [1 - F(x)^\alpha]^{\lambda-1} \quad (4)$$

where α and λ are non-negative shape parameters respectively, where $g(x) = dG(x)/dx$ and for $\alpha, \lambda > 0$ representing additional shape parameters to the distribution G .

2.3 Proposed Kumaraswamy negative exponential distribution (KNED)

The probability density function of the proposed (KNED) is derived by substituting the baseline distribution. Specifically, equations 1 and 2 are substituted into equation 4, respectively. Consequently, the proposed PDF is expressed as follows:

$$g(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} = \frac{\alpha \lambda}{\beta} e^{-\frac{(\varepsilon-x)}{\beta}} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha-1} [1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^\alpha]^{\lambda-1} \quad (5)$$

where $\alpha, \lambda, \beta > 0$.

Also, the cdf of KNED is obtained by substituting equation 2 into equation 3. Thus given as follows:

$$G(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} = 1 - [1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^\alpha]^\lambda \quad (6)$$

where $\alpha, \lambda, \beta > 0$

2.4 Linear expansions for the density function of KNED

Considering the PDF of our proposed model that is eqn (5)

$$g(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} = \frac{\alpha \lambda}{\beta} e^{-\frac{(\varepsilon-x)}{\beta}} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha-1} [1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^\alpha]^{\lambda-1},$$

for $\lambda > 0$ real non-integer, we use the series representation

$$[1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^\alpha]^{\lambda-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\lambda-1}{i} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha i} \quad (7)$$

Substituting eqn(7) into eqn(5), we have

$$g(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} = \frac{\alpha \lambda}{\beta} e^{-\frac{(\varepsilon-x)}{\beta}} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha-1} \sum_{i=0}^{\infty} (-1)^i \binom{\lambda-1}{i} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha i} \quad (8)$$

where the coefficients are

$$w_i = w_i(\alpha, \lambda) = (-1)^i \alpha \lambda \binom{\lambda-1}{i}$$

$$\sum_{i=0}^{\infty} w_i = 1$$

For $\alpha > 0$ real non-integer, we further expand using the series representation to have

$$g(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} = \frac{1}{\beta} e^{-\frac{(\varepsilon-x)}{\beta}} \sum_{i=0}^{\infty} w_i (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha-1} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha i} \quad (9)$$

$$= \frac{1}{\beta} e^{-\frac{(\varepsilon-x)}{\beta}} \sum_{i=0}^{\infty} w_i (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha(i+1)-1} \quad (10)$$

We can expand $(1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha(i+1)-1}$ as follows,

$$(1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha(i+1)-1} = \{1 - [1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})]\}^{\alpha(i+1)-1} \quad (11)$$

Using series expansion we have

$$= \sum_{j=0}^{\infty} (-1)^j \binom{\alpha(i+1)-1}{j} [1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})]^j \quad (12)$$

$$\text{But } [1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})]^j = \sum_{r=0}^j (-1)^r \binom{j}{r} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^r \quad (13)$$

Substituting eqn(13) in eqn(12) we have

$$(1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha(i+1)-1} = \sum_{j=0}^{\infty} \sum_{r=0}^j (-1)^{j+r} \binom{\alpha(i+1)-1}{j} \binom{j}{r} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^r \quad (14)$$

Thus, the density can be rearrange in the form

$$g(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} = \frac{1}{\beta} e^{-\frac{(\varepsilon-x)}{\beta}} \sum_{i=0}^{\infty} w_i \sum_{j=0}^{\infty} \sum_{r=0}^j (-1)^{j+r} \binom{\alpha(i+1)-1}{j} \binom{j}{r} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^r \quad (15)$$

$$= \frac{1}{\beta} e^{-\frac{(\varepsilon-x)}{\beta}} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{r=0}^j w_{i,j,r} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^r \quad (16)$$

where the coefficient

$$w_{i,j,r} = w_{i,j,r}(\alpha, \lambda) = (-1)^{i+j+r} \alpha \lambda \binom{\alpha(i+1)-1}{j} \binom{\lambda-1}{i} \binom{j}{r}$$

$$\text{and } \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{r=0}^j w_{i,j,r} = 1$$

3.0 VALIDITY CHECK OF THE PROPOSED KNED MODEL

This section shows the validation of the proposed KNED.

To demonstrate that the Probability Density Function of the proposed model integrates to unity across its complete range, it is necessary to utilize the attributes of the PDF and fundamental principles of probability. The objective is to establish that the integral of the PDF of the proposed model over the entire range of X equals 1. This implies:

Thus, every distribution should be valid if

$$\int_{-\infty}^{\infty} g(x) dx = 1 \quad (17)$$

As such; we show that

$$\int_0^1 g(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} dx = 1 \quad (18)$$

Substituting eqn(5) into eqn(18), we have

$$\int_0^1 \frac{\alpha \lambda}{\beta} e^{-\frac{(\varepsilon-x)}{\beta}} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha-1} [1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})]^{\lambda-1} dx \quad (19)$$

$$\text{Let } y = 1 - e^{-\frac{(\varepsilon-x)}{\beta}}, \quad \frac{dy}{dx} = \frac{-1}{\beta} e^{-\frac{(\varepsilon-x)}{\beta}} \text{ and } dx = \frac{-\beta dy}{e^{-\frac{(\varepsilon-x)}{\beta}}}$$

Substituting the above terms in eqn(19) we have,

$$\int_0^1 \frac{\alpha\lambda}{\beta} e^{\frac{-(\varepsilon-x)}{\beta}} y^{\alpha-1} [1-y^\alpha]^{\lambda-1} \frac{-\beta dy}{e^{\frac{-(\varepsilon-x)}{\beta}}} \quad (20)$$

$$= \int_0^1 -\alpha\lambda y^{\alpha-1} [1-y^\alpha]^{\lambda-1} dy \quad (21)$$

Let $w = 1 - y^\alpha$, $\frac{dw}{dy} = -\alpha y^{\alpha-1}$ and $dy = \frac{-dw}{\alpha y^{\alpha-1}}$

Substituting the above terms in eqn(21), we have

$$\int_0^1 -\alpha\lambda y^{\alpha-1} w^{\lambda-1} \frac{-dw}{\alpha y^{\alpha-1}} \quad (22)$$

Further simplifying we get
 $= 1 - 0 = 1$.

Thus, we have established that integrating our probability density function across its entire range yields a value of 1. This outcome represents a fundamental characteristic of probability density functions, guaranteeing that the cumulative likelihood of all potential outcomes for a continuous random variable is accounted for and amounts to 1.

4.0 STATISTICAL PROPERTIES OF KNED DISTRIBUTION

Some of the statistical characteristics of KNED, including the quantile function, order statistics, moments, moment generating function, characteristic function, survival function, hazard rate function and cumulative hazard rate will be thoroughly examined.

4.1 Reliability Analysis

Reliability analysis aims to ascertain the degree to which a measurement yields consistent and precise results across various conditions and over time.

Reliability analysis focuses on assessing the dependability and performance of systems, products, or processes over time. It involves the evaluation of the likelihood that a system or component will function without failure within specified conditions for a defined period.

4.1.1 The Survival Function

It is used to describe the likelihood that a specific event (such as malfunction, mortality, or occurrence of a noteworthy event) has not occurred within a specified timeframe or duration.

Mathematically;

$$S(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} = 1 - G(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} \quad (23)$$

Substituting eqn(6) into eqn(23), we have

$$= 1 - \{1 - [1 - (1 - e^{\frac{-(\varepsilon-x)}{\beta}})^\alpha]^\lambda\} \quad (24)$$

$$= [1 - (1 - e^{\frac{-(\varepsilon-x)}{\beta}})^\alpha]^\lambda \quad (25)$$

4.1.2 The Hazard Function

The hazard function quantifies the likelihood of an event occurring within a brief time period, given that the subject has survived until that moment. The hazard function for the specified distribution is formulated as;

$$h(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} = \frac{g(x; \alpha, \lambda, \varepsilon, \beta)_{KNED}}{S(x; \alpha, \lambda, \varepsilon, \beta)_{KNED}} \quad (26)$$

Substituting eqn(5) and eqn(25) into eqn(26), we have

$$= \frac{\frac{\alpha\lambda}{\beta} e^{-\frac{(\varepsilon-x)}{\beta}} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha-1} [1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha}]^{\lambda-1}}{[1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha}]^{\lambda}} \quad (27)$$

Further simplifying we get

$$= \frac{\alpha\lambda}{\beta} e^{-\frac{(\varepsilon-x)}{\beta}} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha-1} [1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha}]^{-1} \quad (28)$$

4.1.3 The Cumulative Hazard Function

The cumulative hazard function offers insights into the total risk of experiencing the event up to a specific time, demonstrating how this risk increases as time advances. This function is derived from the hazard rate function and can be used to estimate the probability of experiencing the event within a given time period.

$$H(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} = -\ln S(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} \quad (29)$$

Substituting eqn(25) into eqn(29), we have

$$= -\ln[1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha}]^{\lambda} \quad (30)$$

4.1.4 The Reverse Hazard Function

Reverse hazard function, also called the reversed hazard rate or the reversed failure rate function, is a concept in survival analysis. It is defined as the probability that an individual survives beyond time t.

$$r(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} = \frac{g(x; \alpha, \lambda, \varepsilon, \beta)_{KNED}}{G(x; \alpha, \lambda, \varepsilon, \beta)_{KNED}} \quad (31)$$

Substituting eqn(5) and eqn(6) into eqn(31), we obtained eqn(32)

$$= \frac{\frac{\alpha\lambda}{\beta} e^{-\frac{(\varepsilon-x)}{\beta}} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha-1} [1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha}]^{\lambda-1}}{1 - [1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha}]^{\lambda}} \quad (32)$$

4.2 The Moment

The moments of a random variable are used to investigate key distribution characteristics, including central tendency, dispersion, and shape. This subsection derives the rth non-central moment of the KNED random variable.

The rth non-central moment of the KNED is defined by,

$$\mu_r^l = E(X^r) = \int_{-\infty}^{\infty} X^r g(x) dx \quad (33)$$

Substituting the eqn(5) into eqn(33), we have

$$= \int_0^1 X^r \frac{\alpha\lambda}{\beta} e^{-\frac{(\varepsilon-x)}{\beta}} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha-1} [1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha}]^{\lambda-1} dx \quad (34)$$

Using power series

$$[1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha}]^{\lambda-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\lambda-1}{i} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha i} \quad (35)$$

Substituting eqn(35) into eqn(34), we have

$$\mu_r^l = \frac{\alpha\lambda}{\beta} \int_0^1 X^r e^{-\frac{(\varepsilon-x)}{\beta}} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha-1} \sum_{i=0}^{\infty} (-1)^i \binom{\lambda-1}{i} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha i} dx \quad (36)$$

$$= \frac{\alpha\lambda}{\beta} \int_0^1 X^r e^{-\frac{(\varepsilon-x)}{\beta}} \sum_{i=0}^{\infty} (-1)^i \binom{\lambda-1}{i} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha(i+1)-1} dx \quad (37)$$

$$\text{Also } (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha(i+1)-1} = \sum_{j=0}^{\infty} \sum_{r=0}^j (-1)^{j+r} \binom{\alpha(i+1)-1}{j} \binom{j}{r} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^r \quad (38)$$

Substituting eqn(38) into eqn(37), we have

$$\mu_r^i = \frac{\alpha\lambda}{\beta} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{r=0}^j (-1)^{i+j+r} \binom{\lambda-1}{i} \binom{\alpha(i+1)-1}{j} \binom{j}{r} \int_0^1 X^r e^{-\frac{(\varepsilon-x)}{\beta}} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^r dx \quad (39)$$

$$\text{Mean} = E(X) = \mu_1, \quad \text{Variance} = \text{Var}(X) = \mu_2 - \mu_1^2,$$

$$\text{Skewness} = \frac{\mu_3 - 3\mu_2\mu_1 + 4\mu_1^3}{(\mu_2 - \mu_1^2)^{\frac{3}{2}}}, \quad \text{Kurtosis} = \frac{\mu_4 - 4\mu_3\mu_1 + 6\mu_1^2\mu_2 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2}$$

4.3 The Moment generating function

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x) dx \quad (40)$$

Using Taylor's series expansion, the moment generating function can be given as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r \quad (41)$$

Where μ_r is the r^{th} non-central moment. Substituting eqn(39) into eqn(41), gives

$$M_X(t) = \frac{\alpha\lambda}{\beta} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{r=0}^j (-1)^{i+j+r} \binom{\lambda-1}{i} \binom{\alpha(i+1)-1}{j} \binom{j}{r} \frac{t^r}{r!} \int_0^1 X^r e^{-\frac{(\varepsilon-x)}{\beta}} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^r dx \quad (42)$$

4.4 The Characteristic function of KNED.

$$\phi_X(t) = E(e^{itx}) = \int_0^{\infty} e^{itx} f(x) dx \quad (43)$$

Using Taylor's series expansion, the characteristic function can be given as

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \mu_r \quad (44)$$

where μ_r is the r^{th} non-central moment. Substituting eqn(39) into eqn(44), gives

$$\phi_X(t) = \frac{\alpha\lambda}{\beta} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{r=0}^j (-1)^{i+j+r} \binom{\lambda-1}{i} \binom{\alpha(i+1)-1}{j} \binom{j}{r} \frac{(it)^r}{r!} \int_0^1 X^r e^{-\frac{(\varepsilon-x)}{\beta}} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^r dx \quad (45)$$

4.5 The Quantile function

The Quantile function of KNED is obtained by inverting the cdf; that is eqn(6) as given below.

$$Q(u) = G^{-1}(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} \quad (46)$$

$$Q(u) = \beta \ln \{ 1 - [1 + (u-1)^{\lambda}]^{-\frac{1}{\alpha}} \} + \varepsilon \quad (47)$$

where $u \sim U(0,1)$.

When we set u at 0.25, 0.5 and 0.75, we obtain the first quartile, the median and the third quartile respectively.

4.6 The ORDER STATISTICS

Order statistics and associated moments possess significant importance in numerous statistical problems and applications within the domains of reliability analysis and life testing. Suppose x_1, x_2, \dots, x_n is a set of random samples from a distribution with pdf, $g(x)$, and let $x_{n1}, x_{n2}, \dots, x_{ni}$ denote the corresponding order statistic obtained from this sample. The i^{th} order statistic of the proposed distributions can be express as:

$$g_{in}(x) = \frac{n!}{(i-1)!(n-i)!} g(x)G(x)^{i-1}(1-G(x))^{n-i} \quad (48)$$

Using power series expression

$$(1-G(x))^{n-i} = \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} G(x)^k \quad (49)$$

Substituting eqn(49) into eqn(48), we have

$$g_{in}(x) = \frac{n!}{(i-1)!(n-i)!} g(x)G(x)^{i-1} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} G(x)^k \quad (50)$$

Further simplifying

$$g_{in}(x) = \frac{n!}{(i-1)!(n-i)!} g(x) \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} G(x)^{i+k-1} \quad (51)$$

Substituting eqn(5) and eqn(6) into eqn(51), we have

$$g_{in}(x) = \frac{n!}{(i-1)!(n-i)!} \left\{ \frac{\alpha\lambda}{\beta} e^{-\frac{(\varepsilon-x)}{\beta}} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha-1} [1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha}]^{\lambda-1} \right\} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \{1 - [1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha}]^{\lambda}\}^{i+k-1} \quad (52)$$

5.0 PARAMETER ESTIMATION USING THE MAXIMUM LIKELIHOOD METHOD

To illustrate how the developed distribution can be applied to model real data sets, it is essential to develop estimators for estimating its parameters. In this section, we will develop estimators for the parameters of the proposed distribution using the widely recognized method of maximum likelihood estimation.

Let x_1, x_2, \dots, x_n be a random sample of size n drawn from the KNED distribution with parameter vector $\theta = (\alpha, \beta, \lambda, \varepsilon)$. The likelihood is defined by:

$$L(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} = \prod_{i=1}^n g(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} \quad (53)$$

Substituting eqn(5) into eqn(53), we have

$$= \prod_{i=1}^n \left\{ \frac{\alpha\lambda}{\beta} e^{-\frac{(\varepsilon-x)}{\beta}} (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha-1} [1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha}]^{\lambda-1} \right\} \quad (54)$$

$$= \left\{ \frac{\alpha\lambda}{\beta} \right\}^n \prod_{i=1}^n e^{-\frac{(\varepsilon-x)}{\beta}} \prod_{i=1}^n (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha-1} \prod_{i=1}^n [1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha}]^{\lambda-1} \quad (55)$$

the log likelihood of KNED distribution will be:

$$\ell(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} = \log L(x; \alpha, \lambda, \varepsilon, \beta)_{KNED} \quad (56)$$

Substituting eqn(55) into eqn(56), we have

$$= \log \left\{ \left(\frac{\alpha\lambda}{\beta} \right)^n \prod_{i=1}^n e^{-\frac{(\varepsilon-x)}{\beta}} \prod_{i=1}^n (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha-1} \prod_{i=1}^n [1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha}]^{\lambda-1} \right\} \quad (57)$$

$$= n \log \alpha + n \log \lambda - n \log \beta + \sum_{i=1}^n \log(e^{-\frac{(\varepsilon-x)}{\beta}}) + (\alpha-1) \sum_{i=1}^n \log(1 - e^{-\frac{(\varepsilon-x)}{\beta}}) + (\lambda-1) \sum_{i=1}^n \log[1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha}] \quad (58)$$

The MLE of the parameters α, β, λ and ε denoted by $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$ and $\hat{\varepsilon}$ are obtained by differentiating eqn(58) partially with respect to α, β, λ and ε . Hence, the components of the unit score vector $\mu\theta = (\mu_{\alpha}, \mu_{\lambda}, \mu_{\beta}, \text{and } \mu_{\varepsilon})'$ are obtained below:

$$\mu_{\alpha} = \frac{\partial \ell(x; \alpha, \lambda, \varepsilon, \beta)_{KNED}}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log e^{-\frac{(\varepsilon-x)}{\beta}} \left\{ 1 - \frac{(\lambda-1)(1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha}}{1 - (1 - e^{-\frac{(\varepsilon-x)}{\beta}})^{\alpha}} \right\} \quad (59)$$

$$\mu_{\lambda} = \frac{\partial \ell(x; \alpha, \lambda, \varepsilon, \beta)_{KNED}}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^n \log \left\{ 1 - \left(1 - e^{-\frac{\varepsilon-x}{\beta}} \right)^{\alpha} \right\} \quad (60)$$

$$\mu_{\beta} = \frac{\partial \ell(x; \alpha, \lambda, \varepsilon, \beta)_{KNED}}{\partial \beta} = \sum_{i=1}^n \left[\frac{2(\varepsilon-x)}{\beta^2} - \frac{2(\varepsilon-x)e^{-\frac{\varepsilon-x}{\beta}}}{\beta(1-e^{-\frac{\varepsilon-x}{\beta}})} \left\{ 1 - \frac{\alpha(\lambda-1)}{(1-e^{-\frac{\varepsilon-x}{\beta}})^{-\alpha} - 1} \right\} \right] \quad (61)$$

$$\mu_{\varepsilon} = \frac{\partial \ell(x; \alpha, \lambda, \varepsilon, \beta)_{KNED}}{\partial \varepsilon} = \sum_{i=1}^n \left[\frac{e^{-\frac{\varepsilon-x}{\beta}}}{\beta(1-e^{-\frac{\varepsilon-x}{\beta}})} - \frac{1}{\beta} \left\{ 1 - \frac{\alpha(\lambda-1)}{(1-e^{-\frac{\varepsilon-x}{\beta}})^{-\alpha} - 1} \right\} \right] \quad (62)$$

The log-likelihood function of this distribution is challenging to solve analytically due to its complex form. However, it can be maximized by employing global optimization methods available in software packages such as R, SAS, and python.

6.0 APPLICATIONS

The flexibility of Kumaraswamy Negative exponential distribution will be illustrated by fitting the model to a life data. We considered the data set corresponding to breaking stress of carbon fibers of 100 observations as discussed by (Shams *et al.* 2013). The data are:

3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 1.12, 3.27, 2.87, 1.47, 3.11, 1.84, 0.39, 1.71, 2.88, 4.42, 2.41, 3.19, 3.22, 1.69, 2.03, 3.28, 3.09, 1.87, 3.15, 4.9, 3.68, 1.61, 2.48, 2.82, 3.75, 2.43, 2.95, 2.97, 1.69, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 4.38, 0.85, 1.61, 2.05, 3.39, 2.81, 4.2, 2.17, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 1.17, 3.56, 2.79, 4.7, 3.65, 3.15, 2.35, 5.08, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.48, 2.83, 1.92, 2.03, 1.8, 1.89, 1.41, 1.18, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 1.25, 3.68, 1.84, 1.59, 1.57, 1.08, 2.12, 3.51, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.17, 2, 1.22.

The maximum likelihood estimates (MLE) of the model parameters are calculated, and various goodness-of-fit statistics for the distributions are compared. The analytical measures of goodness-of-fit, including the Mean Squared Error (MSE), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), and Hanna-Quinn Information Criterion (HQIC), was considered to compare the proposed model with the baseline and other extensions of exponential distributions. In general, a model with lower values of these analytical measures indicates a better fit to the data. All the required computations have been performed in the Python programming language.

CDF Plot

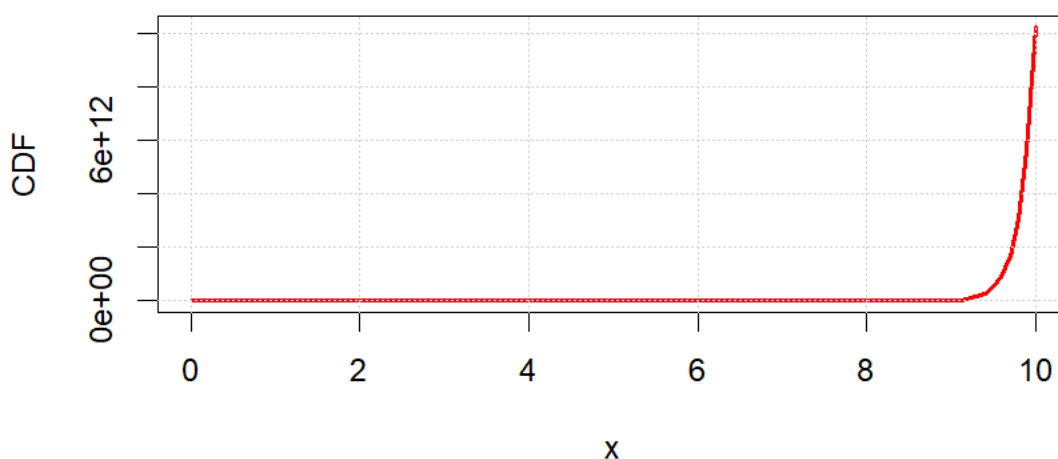


Figure 1: The cumulative distribution function plot of KNED.

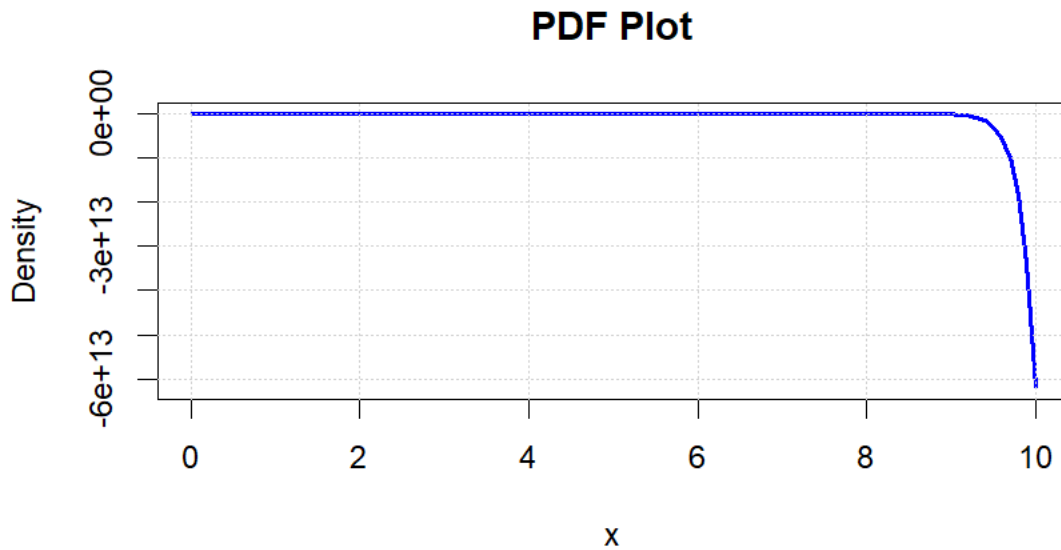


Figure 2: The probability density function plot of KNED

Table 1: The maximum likelihood estimates and their standard error in parenthesis

Parameters					
Models	ϵ	β	α	λ	τ
ED	-	0.3815(0.1265)	-	-	-
NED	6.0(0.0036)	1.0(0.0658)	-	-	-
EED	-	1.0132(0.0633)	-	-	7.7882(0.4304)
KWEED	-	0.1110(0.1126)	0.0660(0.2376)	66.246(0.662)	4.9502(0.0125)
KNED	5.8964(0.0026)	2.1473(0.0566)	2.2777(0.2365)	1.9572(0.0325)	-

Table 2: The analytical measures of the models fit

Models	LL	KS	AD	AIC	BIC	CAIC	HQIC
ED	-196.371	0.038	0.263	394.742	397.397	394.783	390.381
NED	-192.722	0.078	0.886	293.562	323.923	320.603	324.336
EED	-146.182	0.048	0.324	296.365	301.575	296.488	292.861
KWEED	-141.318	0.042	0.271	290.637	301.057	291.058	293.213
KNED	-139.821	0.031	0.223	286.367	301.005	290.506	291.665

Table 2 provides a comparative analysis of the log-likelihood (LL), Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC) for the Kumaraswamy negative exponential distribution (KNED), Kumaraswamy generalized exponentiated exponential distribution (KwGEED), negative exponential distribution (NED), exponentiated exponential distribution (EED), and exponential

distribution (ED). The results indicate that the Kumaraswamy negative exponential distribution has lower values for these information criteria, suggesting it is a better fit for the data.

7.0 CONCLUSION

This study developed a four-parameter continuous probability distribution model known as the Kumaraswamy negative exponential distribution (KNED). The probability density function and

cumulative distribution function of this constructed distribution were derived. Additionally, the validity of the modified distribution was demonstrated. Various statistical properties of the proposed distribution, including reliability analysis, moments, moment-generating functions, and order statistics, have been presented. The method of maximum likelihood was employed to estimate the model parameters. Some analytical measures such as AIC, BIC, CAIC and HQIC were used in assessment and comparison of the modified model to other variations of exponential distribution and the resulting distribution was found to be a better fit for modelling real-world data.

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