

Common L -Fuzzy Fixed Point Results for Contractive L -Fuzzy Maps in \mathcal{F} -Metric Space

Muhammad Sirajo Abdullahi and Jamilu Abubakar

Department of Mathematics, Usmanu Danfodiyo University, Sokoto, Nigeria

Corresponding Author's Email: abubakar.jamilu@udusok.edu.ng

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ABSTRACT

In this paper, we establish an L -fuzzy fixed point theorem for generalized contractive L -fuzzy mappings in the context of an \mathcal{F} -complete \mathcal{F} -metric space. We also provide consequences and examples to support the main findings.

Keywords: Fixed point; common fixed point; L -fuzzy fixed point; common L -fuzzy fixed point; L -fuzzy mapping.

1.0 INTRODUCTION

Zadeh's (1965) work on fuzzy set theory in 1965 significantly simplified the approach to solving real-world problems by offering a clearer and more precise method for handling uncertainties and vagueness. Two years later, his student, (Goguen, 1967), expanded this concept to L -fuzzy set theory by considering a completely distributive lattice L instead of the interval $[0, 1]$, thereby broadening the scope and applicability of fuzzy sets.

In 1981, (Heilpern, 1981) introduced fuzzy contraction mappings, generalizing both the Banach contraction principle and Nadler's fixed point theorems, leading to a fuzzy fixed point result in complete metric spaces. Frigon & O'Regan (2002) subsequently expanded this work by examining contractive conditions for 1-level sets of fuzzy contractions, inspiring further generalizations. In 2014, (Rashid et al., 2014) introduced the concept of β_{FL} -admissibility for L -fuzzy mappings and proved a common L -fuzzy fixed point theorem, thereby advancing the field with the introduction of d_L^∞ -metrics and Hausdorff distances for L -fuzzy sets.

In 2018, the notion of an \mathcal{F} -metric space was introduced by (Jleli & Samet, 2018), this concept generalized the classical metric space. Later, (Alnaser et al., 2019) applied this framework to establish fixed point theorems

for contractions involving rational expressions, while (Alansari et al., 2020) explored further results in \mathcal{F} -metric spaces and identified open problems, including common fixed point theorems for L -fuzzy mappings in such spaces. Recently, (Lateef, 2024) addressed this problem by establishing common L -fuzzy fixed point theorems for $(\beta - \psi)$ contractions in \mathcal{F} -metric spaces, further advancing the field. In line with this, we offer another solution to the problem via generalized contraction-type mappings without using admissible mappings. For more on \mathcal{F} -metric spaces and fuzzy fixed point theory, see (Abdullahi et al., 2018, 2021, 2024; Abdullahi & Azam, 2017, 2021; Al-Mezel et al., 2020; Asif et al., 2019; Azam & Beg, 2013; Gopal et al., 2021; Hussain & Kanwal, 2018; Kanwal et al., 2022; Khan et al., 2023; Lateef & Ahmad, 2019; Manav & Turkoglu, 2019; Martinez-Moreno & Kumam, 2016; Mohammed et al., 2021; Mohammed & Azam, 2019; Phiangsungnoen et al., 2014; Rakić et al., 2020; Shagari & Azam, 2020; Zahed et al., 2023) along with references therein.

In this paper, we develop a new L -fuzzy common fixed point theorem for generalized contractive L -fuzzy mappings within the framework of \mathcal{F} -complete \mathcal{F} -metric spaces. Notably, our work does not focus on admissible mappings, which distinguishes it from some previous research. Additionally, we explore the consequences of our results and provide examples to support our findings.

2.0 PRELIMINARIES

We begin the section by recalling the notion of a metric space as defined below:

Definition 2.1. A metric space is a pair (J, d) where J is a nonempty set and $d : J \times J \rightarrow [0, \infty)$ is a function satisfying the following properties, for all $j, r, v \in J$:

- 1) *Non-negativity:* $d(j, r) \geq 0$.
- 2) *Identity of indiscernible:* $d(j, r) = 0$ if and only if $j = r$.
- 3) *Symmetry:* $d(j, r) = d(r, j)$.
- 4) *Triangle inequality:* $d(j, v) \leq d(j, r) + d(r, v)$.

In 2018, (Jleli & Samet, 2018) introduced a remarkable generalization of metric space, known as an \mathcal{F} -metric space, defined as follows:

Let \mathcal{F} denote the set of functions $f: (0, +\infty) \rightarrow \mathbb{R}$ satisfying the following conditions:

($\mathcal{F}1$) If $0 < b < a$, then $f(b) \leq f(a)$.

($\mathcal{F}2$) For any sequence $\{k_n\} \subset (0, +\infty)$,

$$\lim_{n \rightarrow \infty} k_n = 0 \Leftrightarrow \lim_{n \rightarrow \infty} f(k) = -\infty.$$

Definition 2.2. Let J be a nonempty set and $dF: J \times J \rightarrow [0, \infty)$ be a given function. Suppose that there exists $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$ such that for all $j, r, v \in J$:

1) *Non-negativity*: $dF(j, r) \geq 0$.

2) *Identity of indiscernible*: $d(j, r) = 0$ if and only if $j = r$.

3) *Symmetry*: $dF(j, r) = dF(r, j)$.

4) *\mathcal{F} -metric condition*: for every $n^* \in \mathbb{N}$ with $n^* \geq 2$, and every sequence $\{p_i\}_{i=1}^{n^*} \subset J$ with $(x_1, x_{n^*}) = (j, r)$, we have:

$$dF(j, r) > 0 \Rightarrow f(dF(j, r)) \leq f\left(\sum_{i=1}^{n^*-1} dF(p_i, p_{i+1})\right) + \alpha.$$

Then, the function dF is called an \mathcal{F} -metric and the pair (J, dF) an \mathcal{F} -metric space.

Remark 2.3. Observe that any metric d on J is an \mathcal{F} -metric dF on J . This is because the function $f(\pi) = \ln \pi$ for $\pi > 0$ satisfies the \mathcal{F} -metric condition with $\alpha = 0$. Thus, for any metric d , we have:

$$dF(j, r) > 0 \Rightarrow f(dF(j, r)) \leq f\left(\sum_{i=1}^{n^*-1} dF(p_i, p_{i+1})\right).$$

Example 2.4. (Jleli & Samet, 2018). The function $dF: \mathbb{R} \times \mathbb{R} \rightarrow [0, +\infty)$ defined as:

$$dF(j, r) = \begin{cases} |j - r| & \text{if } (j, r) \in [0, 3] \times [0, 3], \\ |j - r|^2 & \text{if } (j, r) \in [0, 3] \times [0, 3], \end{cases}$$

with $f(\pi) = \ln \pi$ for all $\pi > 0$ and $\alpha = \ln 3$, is an \mathcal{F} -metric.

Example 2.5. Let d be the standard metric on \mathbb{R}^n defined by:

$$dF(j, r) = \|j - r\|$$

where $\|\cdot\|$ is the Euclidean norm, $\alpha = 0$, and let f be given as $f(t) = \pi^p$ for all $\pi > 0$ and for some $p \geq 1$. Then dF is an \mathcal{F} -metric.

The structure of \mathcal{F} -metric (dF) allows for the study of convergence, continuity, and compactness in a more generalized setting compared to traditional metric spaces.

Definition 2.6. (Jleli & Samet, 2018). Let (J, dF) be an \mathcal{F} -metric space.

(i) A sequence $\{j_n\} \in J$ is called \mathcal{F} -convergent to $j \in J$ if $\{j_n\}$ converges to j with respect to the \mathcal{F} -metric dF .

(ii) The sequence $\{j_n\}$ is said to be \mathcal{F} -Cauchy if

$$\lim_{n, m \rightarrow \infty} dF(j_n, j_m) = 0.$$

(iii) The \mathcal{F} -metric space (J, dF) is called \mathcal{F} -complete if every \mathcal{F} -Cauchy sequence in J is \mathcal{F} -convergent to a point in J .

Definition 2.7. A fuzzy set in J is a function with domain J and range in $[0, 1]$. I.e., S is a fuzzy set if $S: J \rightarrow [0, 1]$.

Let $\mathcal{F}(J)$ denote the collection of all fuzzy subsets of J . For a fuzzy set S , $S(j)$ is termed the grade of membership of j in S where $j \in J$. The α -level set of S denoted by $[S]_\alpha$ is defined as follows:

$$[S]_\alpha = \{j \in J : S(j) \geq \alpha\}, \text{ for } \alpha \in (0, 1],$$

$$[S]_0 = \text{closure of the set } \{j \in J : S(j) > 0\}.$$

Define $\mathcal{J}(J) \subset \mathcal{F}(J)$ as in the following:

$$\mathcal{J}(J) = \{S \in \mathcal{F}(J) : [S]_\alpha \text{ is compact and nonempty, } \alpha \in (0, 1]\}.$$

Definition 2.8. (Abdullahi & Azam, 2017). Let (L, \leq_L) be a partially ordered set, then it is termed as a:

i. *lattice*; if $p \vee q \in L, p \wedge q \in L$ for any $p, q \in L$,

ii. *complete lattice*; if $\bigvee P \in L, \bigwedge P \in L$ for any $P \in L$,

iii. *distributive lattice*; if $p \vee (q \wedge x) = (p \vee q) \wedge (p \vee x)$, and $p \wedge (q \vee x) = (p \wedge q) \vee (p \wedge x)$ for any $p, q \in L$,

iv. *complete distributive lattice*; if $p \vee (\bigwedge q_i) = \bigwedge (p \vee q_i)$, and $p \wedge (\bigvee q_i) = \bigvee (p \wedge q_i)$ for any $p, q_i \in L$,

v. *bounded lattice*; if in addition to being lattice it has both top (1_L) and bottom (0_L) elements, that satisfy the following condition: $0_L \leq_L l \leq_L 1_L$ for every $l \in L$.

Definition 2.9. (Abdullahi & Azam, 2017; Goguen, 1967). Let J be a nonempty set. An L -fuzzy set S on J is a function $S: J \rightarrow L$, where L is bounded complete distributive lattice with top and bottom elements.

Definition 2.10. (Goguen, 1967). Let L be a lattice, with 1_L as its top element and 0_L as its bottom element. Suppose that for $p, q \in L$, we have $p \vee q = 1_L$ and $p \wedge q = 0_L$ then q is a unique complement of p which is usually denoted as \check{p} .

Remark 2.11. (Abdullahi & Azam, 2017). If $L = [0, 1]$, the L -fuzzy set becomes a specific case of fuzzy sets (Zadeh, 1965), indicating that L -fuzzy sets are a more general concept.

Throughout, we let dF denote an \mathcal{F} -metric, so that (J, dF) represents an \mathcal{F} -metric space. Let $\mathcal{B}(J)$ denote the set of nonempty, closed and bounded subsets of J , and $\mathcal{D}(J)$ denote the set of nonempty compact subsets of J . Let $U, Z \in \mathcal{B}(J)$, then we define

$$p_{\alpha_L}^F(j, U) = \inf_{r \in U_{\alpha_L}} dF(j, r),$$

$$p_{\alpha_L}^F(U, Z) = \inf_{j \in U_{\alpha_L}, r \in Z_{\alpha_L}} dF(j, r),$$

and $p_F(U, Z) = \sup_{\alpha_L} p_{\alpha_L}^F(U, Z)$. Then, the Hausdorff metric H_F on $\mathcal{D}(J)$ induced by the \mathcal{F} -metric dF is defined as:

$$H_F(U_{\alpha_L}, Z_{\alpha_L}) = \max \left\{ \sup_{j \in U_{\alpha_L}} dF(j, Z_{\alpha_L}), \sup_{r \in Z_{\alpha_L}} dF(r, U_{\alpha_L}) \right\}.$$

Let $\mathcal{F}_L(J)$ denote the class of all L -fuzzy subsets of J . Let $\mathcal{K}_L(J) \subset \mathcal{F}_L(J)$ be defined as in the following:

$$\mathcal{K}_L(J) = \{S \in \mathcal{F}_L(J) : S_{\alpha_L} \text{ is compact and nonempty, where } \alpha_L \in L \setminus \{0_L\}\}.$$

We define and denote the α_L -level set of an L -fuzzy set S as in the following:

$$S_{\alpha_L} = \{j \in J : \alpha_L \leq_L S(j)\} \text{ for } \alpha_L \in L \setminus \{0_L\},$$

$$S_{0_L} = \text{the closure of the set } \{j \in J : 0_L \leq_L S(j)\}.$$

Let $U, Z \in \mathcal{F}_L(J)$, $U \subset Z \Leftrightarrow U(j) \leq_L Z(j)$ for every $j \in J$. Suppose that there is an $\alpha_L \in L \setminus \{0_L\}$ so that the α_L -level sets of the L -fuzzy sets U and Z are closed and bounded, i.e., $U_{\alpha_L}, Z_{\alpha_L} \in \mathcal{B}(J)$. Then,

$$D_{\alpha_L}^F(U, Z) = H_F(U_{\alpha_L}, Z_{\alpha_L}).$$

For every $\alpha_L \in L \setminus \{0_L\}$, let $U_{\alpha_L}, Z_{\alpha_L} \in \mathcal{B}(J)$. Then,

$$dF_L^\infty(U, Z) = \sup_{\alpha_L} D_{\alpha_L}^F(U, Z).$$

It is not difficult to see that, dF_L^∞ is an \mathcal{F} -metric on $\mathcal{F}_L(J)$ and that the completeness of the \mathcal{F} -metric space (J, dF) implies that $(\mathcal{D}(J), H_F)$ and $(\mathcal{F}_L(J), dF_L^\infty)$ are \mathcal{F} -complete.

Definition 2.12. Let J be a crisp set and R be an \mathcal{F} -metric space. If $Q : J \rightarrow \mathcal{F}_L(R)$, then Q is termed an L -fuzzy mapping, which is subset on $J \times R$ with membership function $Q(j)(r)$. The function $Q(j)(r)$ is the grade of membership of r in $T(j)$.

Definition 2.13. Let (J, dF) be an \mathcal{F} -metric space and $I, Q : J \rightarrow \mathcal{F}_L(J)$. A point $v \in J$ is termed common L -fuzzy fixed point of I and Q if $v \in [Iv]_{\alpha_L} \cap [Qv]_{\alpha_L}$, for some $\alpha_L \in L \setminus \{0_L\}$.

Remark 2.14. If $\alpha_L = 1_L$, then the common L -fuzzy fixed point becomes the specific case termed common fixed point of the L -fuzzy mappings I and Q , indicating that L -fuzzy sets are a more general concept.

3.0 COMMON L -FUZZY FIXED POINT THEOREMS

In the sequel, we will recall some important results necessary for our findings before stating the theorem.

Lemma 3.1. (Berinde, 1997). Let $\chi \in \Omega$. Then, the following properties hold:

- (i) For $k \geq 1$, each iterate χ^k of χ is a comparison function.
- (ii) $\chi(\pi) < \pi$ for all $\pi > 0$.
- (iii) χ is continuous at 0.

Lemma 3.2. (Alansari et al., 2020). Let (J, dF) be an \mathcal{F} -metric space. Let U and Z be nonempty, closed and compact subsets of J . Then for any $u \in U$ there exists $z \in Z$ such that $dF(u, z) \leq H_F(U, Z)$.

We are now ready to establish our common L -fuzzy fixed point existence result for some generalized contractive L -fuzzy functions in the context of \mathcal{F} -complete \mathcal{F} -metric spaces.

Theorem 3.3. Let (J, dF) be an \mathcal{F} -complete \mathcal{F} -metric space and $I, Q : J \rightarrow \mathcal{K}_L(J)$ be L -fuzzy functions. Assume that for every $j \in J$, there exist $\alpha_L \in L \setminus \{0_L\}$ such that $[Ij]_{\alpha_L}, [Qj]_{\alpha_L} \in \mathcal{D}(J)$. Suppose that the following holds:

$$D_{\alpha_L}^F(Ij, Qr) \leq \chi(\Gamma(j, r)) \quad (3.1)$$

where $\chi \in \Omega$ and

$$\Gamma(j, r) = \max \left\{ dF(j, r), p_{\alpha_L}^F(j, Ij), p_{\alpha_L}^F(r, Qr), \frac{p_{\alpha_L}^F(j, Ij) + p_{\alpha_L}^F(r, Qr)}{2} \right\},$$

for all $j, r \in J$. Then, the functions I and Q have at least a common L -fuzzy fixed point.

Proof: Let $j_0 \in J$ and $j_1 \in [Ij_0]_{\alpha_L}$. By Lemma 3.2 and the fact that $[Ij_0]_{\alpha_L}$ is nonempty and compact subset of J , then there exists $j_2 \in [Qj_1]_{\alpha_L}$, such that

$$dF(j_1, j_2) = p_{\alpha_L}^F(j_1, Qj_1) \leq H_F([Ij_0]_{\alpha_L}, [Qj_1]_{\alpha_L}) = D_{\alpha_L}^F(Ij_0, Qj_1). \quad (3.2)$$

By (3.1) and (3.2), we have

$$dF(j_1, j_2) \leq D_{\alpha_L}^F(Ij_0, Qj_1) \leq \chi(\Gamma(j_0, j_1)) \quad (3.3)$$

where

$$\begin{aligned} \Gamma(j_0, j_1) &\leq \max \left\{ dF(j_0, j_1), p_{\alpha_L}^F(j_0, Ij_0), (j_1, Qj_1), \frac{p_{\alpha_L}^F(j_0, Ij_0) + p_{\alpha_L}^F(j_1, Qj_1)}{2} \right\} \\ &= \max \left\{ dF(j_0, j_1), p_{\alpha_L}^F(j_0, j_1), p_{\alpha_L}^F(j_1, j_2), \frac{p_{\alpha_L}^F(j_0, j_1) + p_{\alpha_L}^F(j_1, j_2)}{2} \right\} \\ &\leq \max \{ dF(j_0, j_1), p_{\alpha_L}^F(j_0, j_1), p_{\alpha_L}^F(j_1, j_2) \} \end{aligned}$$

Suppose that $p_{\alpha_L}^F(j_1, j_2)$ is the maximum, then we have

$$dF(j_1, j_2) \leq \chi(p_{\alpha_L}^F(j_1, j_2)) < p_{\alpha_L}^F(j_1, j_2) \leq dF(j_1, j_2)$$

which is a contradiction. Hence $dF(j_0, j_1)$ has to be the maximum. Now, equation (3.3) becomes

$$dF(j_1, j_2) \leq \chi(dF(j_0, j_1)).$$

Similarly, continuous repetition of the process above will yield a sequence $\{j_n\}_{n \in \mathbb{N}}$ of elements of J such that $j_{2n-1} \in [Ij_{2n-2}]_{\alpha_L}$ and $j_{2n} \in [Qj_{2n-1}]_{\alpha_L}$. Consequently,

$$dF(j_{2n}, j_{2n+1}) \leq \chi(dF(j_{2n-1}, j_{2n})),$$

and

$$dF(j_{2n-1}, j_{2n}) \leq \chi(dF(j_{2n-2}, j_{2n-1})).$$

Applying induction, for all $n \in \mathbb{N}$, we get

$$\begin{aligned} dF(j_n, j_{n+1}) &\leq \chi(dF(j_{n-1}, j_n)) \\ &\leq \chi(\chi(dF(j_{n-2}, j_{n-1}))) \\ &\vdots \\ &\leq \chi^n(dF(j_0, j_1)). \end{aligned}$$

Now, we next show that the sequence $\{j_n\}_{n \in \mathbb{N}}$ is an \mathcal{F} -Cauchy sequence in J . Let $\psi > 0$ be a given positive number and $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that the \mathcal{F} -metric condition is satisfied. By condition (F2), there exists $\pi > 0$ such that

$$0 < s < \pi \Rightarrow f(s) \leq f(\pi) - \alpha. \quad (3.4)$$

Let $N_0 \in \mathbb{N}$ such that

$$0 < \sum_{n \geq N_0} \chi^n dF(j_0, j_1) < \pi. \quad (3.5)$$

Now, from (3.4), (3.5) and condition (F2), we have

$$f\left(\sum_{i=n}^{m-1} \chi^i dF(j_0, j_1)\right) \leq f\left(\sum_{n \geq N_0} \chi^n dF(j_0, j_1)\right) \leq f(\psi) - \alpha, \quad (3.6)$$

for $m > n \geq N_0$. So, using (3.6) and the \mathcal{F} -metric condition, we have $dF(j_n, j_m) > 0$. Subsequently,

$$f(dF(j_n, j_m)) \leq f\left(\sum_{i=n}^{m-1} dF(j_i, j_{i+1})\right) + \alpha \leq f(\psi).$$

It follows from condition (F1) that

$$dF(j_n, j_m) < \psi, \quad m > n \geq N_0.$$

This shows that $\{j_n\}_{n \in \mathbb{N}}$ is an \mathcal{F} -Cauchy sequence. Hence, the \mathcal{F} -completeness of (J, dF) implies that there exists $r^* \in J$ such that $j_n \rightarrow r^*$ as $n \rightarrow \infty$.

Now, to prove that $r^* \in [Ir^*]_{\alpha_L}$. We assume for contradiction that $p_{\alpha_L}^F(r^*, Ir^*) > 0$. Then, by the \mathcal{F} -metric condition, we get

$$Q(j)(\pi) = \begin{cases} \beta, & \text{if } 0 \leq \pi \leq \frac{j}{12}; \\ \eta, & \text{if } \frac{j}{12} < \pi \leq \frac{j}{8}; \\ \omega, & \text{if } \frac{j}{8} < \pi \leq \frac{j}{4}; \\ \tau, & \text{if } \frac{j}{4} < \pi \leq 1. \end{cases}$$

For every $j \in J$, there exists $\alpha_L = \beta$ for which

$$[Ij]_{\beta} = \left[0, \frac{j}{6}\right] \quad \text{and} \quad [Qj]_{\beta} = \left[0, \frac{j}{12}\right].$$

For each $j, r \in J$, we have

$$\begin{aligned} D_{\beta}^F(Ij, Qr) &\leq H_F([Ij]_{\beta}, [Qr]_{\beta}) \\ &= H_F\left(\left[0, \frac{j}{6}\right], \left[0, \frac{r}{12}\right]\right) \\ &= \frac{1}{12}|j - r| \\ &< \frac{1}{6}|j - r| \\ &= \chi|j - r| \\ &\leq \chi \left(\max \left\{ |j - r|, \left| j - \frac{j}{6} \right|, \left| r - \frac{r}{12} \right| \right\} \right) \\ &= \chi \left(\max \left\{ dF(j, r), p_{\beta}^F(j, Ij), p_{\beta}^F(r, Qr), \frac{p_{\beta}^F(j, Ij) + p_{\beta}^F(r, Qr)}{2} \right\} \right), \end{aligned}$$

where $\chi(\pi) = \pi$ for all $\pi > 0$. It is not difficult to see that all the conditions of Theorem 3.3 are satisfied. Therefore, there exists an element $0 \in J$ so that $0 \in [I0]_{\beta} \cap [Q0]_{\beta}$.

As consequences of the theorem above, we provide the following important corollaries.

Corollary 3.5. Let (J, dF) be an \mathcal{F} -complete \mathcal{F} -metric space and $I, Q : J \rightarrow \mathcal{K}_L(J)$ be L -fuzzy functions. Assume that for every $j \in J$, there exist $\alpha_L \in L \setminus \{0_L\}$ such that $[Ij]_{\alpha_L}, [Qj]_{\alpha_L} \in \mathcal{D}(J)$. Suppose that for all $j, r \in J$, the following holds:

$$D_{\alpha_L}^F(Ij, Qr) \leq \chi(\max\{dF(j, r), p_{\alpha_L}^F(j, Ij), p_{\alpha_L}^F(r, Qr)\})$$

where $\chi \in \Omega$. Then, the functions I and Q have at least a common L -fuzzy fixed point.

Corollary 3.6. Let (J, dF) be an \mathcal{F} -complete \mathcal{F} -metric space and $Q : J \rightarrow \mathcal{K}_L(J)$ be L -fuzzy function. Assume that for every $j \in J$, there exist $\alpha_L \in L \setminus \{0_L\}$ such that $[Qj]_{\alpha_L} \in \mathcal{D}(J)$. Suppose that the following holds:

$$D_{\alpha_L}^F(Qj, Qr) \leq \chi(\Gamma(j, r))$$

where $\chi \in \Omega$ and

$$\Gamma(j, r) = \max \left\{ dF(j, r), p_{\alpha_L}^F(j, Qj), p_{\alpha_L}^F(r, Qr), \frac{p_{\alpha_L}^F(j, Qj) + p_{\alpha_L}^F(r, Qr)}{2} \right\}$$

for all $j, r \in J$. Then, the function Q has at least an L -fuzzy fixed point.

Corollary 3.7. Let (J, dF) be an \mathcal{F} -complete \mathcal{F} -metric space and $Q : J \rightarrow \mathcal{K}_L(J)$ be L -fuzzy function. Assume that for every $j \in J$, there exist $\alpha_L \in L \setminus \{0_L\}$ such that $[Qj]_{\alpha_L} \in \mathcal{D}(J)$. Suppose that for all $j, r \in J$, the following holds:

$$D_{\alpha_L}^F(Qj, Qr) \leq \chi(\max\{dF(j, r), p_{\alpha_L}^F(j, Qj), p_{\alpha_L}^F(r, Qr)\})$$

where $\chi \in \Omega$. Then, the function Q has at least an L -fuzzy fixed point.

Corollary 3.8. (Alansari et al., 2020). Let (J, dF) be an \mathcal{F} -complete \mathcal{F} -metric space and $I, Q : J \rightarrow \mathcal{J}(J)$ be fuzzy functions. Assume that for every $j \in J$, there exist $\alpha_I(j), \alpha_Q(j) \in (0, 1]$ such that $[Ij]_{\alpha_I(j)}, [Qj]_{\alpha_Q(j)} \in \mathcal{D}(J)$. Suppose that the following holds:

$$H_F([Ij]_{\alpha_I(j)}, [Qr]_{\alpha_Q(r)}) \leq \chi(\Gamma(j, r))$$

where $\chi \in \Omega$ and

$$\Gamma(j, r) = \max \left\{ dF(j, r), dF(j, [Ij]_{\alpha_I(j)}), dF(r, [Qr]_{\alpha_Q(r)}), \frac{dF(j, [Ij]_{\alpha_I(j)}) + dF(r, [Qr]_{\alpha_Q(r)})}{2} \right\},$$

for all $j, r \in J$. Then, the functions I and Q have at least a common fuzzy fixed point.

Corollary 3.9. Let (J, dF) be an \mathcal{F} -complete \mathcal{F} -metric space and $I, Q : J \rightarrow \mathcal{J}(J)$ be fuzzy functions. Assume that for every $j \in J$, there exist $\alpha_I(j), \alpha_Q(j) \in (0, 1]$ such that $[Ij]_{\alpha_I(j)}, [Qj]_{\alpha_Q(j)} \in \mathcal{D}(J)$. Suppose that for all $j, r \in J$, the following holds:

$$H_F([Ij]_{\alpha_I(j)}, [Qr]_{\alpha_Q(r)}) \leq \chi \left(\max \left\{ dF(j, r), dF(j, [Ij]_{\alpha_I(j)}), dF(r, [Qr]_{\alpha_Q(r)}) \right\} \right)$$

where $\chi \in \Omega$. Then, the functions I and Q have at least a common fuzzy fixed point.

Corollary 3.10. Let (J, dF) be an \mathcal{F} -complete \mathcal{F} -metric space and $Q : J \rightarrow \mathcal{J}(J)$ be a fuzzy function. Assume that for every $j \in J$, there exists $\alpha_Q(j) \in (0, 1]$ such that $[Qj]_{\alpha_Q(j)} \in \mathcal{D}(J)$. Suppose that the following holds:

$$H_F([Qj]_{\alpha_Q(j)}, [Qr]_{\alpha_Q(r)}) \leq \chi(\Gamma(j, r))$$

where $\chi \in \Omega$ and

$$\Gamma(j, r) = \max \left\{ dF(j, r), dF(j, [Qj]_{\alpha_Q(j)}), dF(r, [Qr]_{\alpha_Q(r)}), \frac{dF(j, [Qj]_{\alpha_Q(j)}) + dF(r, [Qr]_{\alpha_Q(r)})}{2} \right\},$$

for all $j, r \in J$. Then, the function Q has at least a fuzzy fixed point.

Corollary 3.11. Let (J, dF) be an \mathcal{F} -complete \mathcal{F} -metric space and $Q : J \rightarrow \mathcal{J}(J)$ be fuzzy functions. Assume that for every $j \in J$, there exists $\alpha_Q(j) \in (0, 1]$ such that $[Qj]_{\alpha_Q(j)} \in \mathcal{D}(J)$. Suppose that for all $j, r \in J$, the following holds:

$$H_F([Qj]_{\alpha_Q(j)}, [Qr]_{\alpha_Q(r)}) \leq \chi \left(\max \left\{ dF(j, r), dF(j, [Qj]_{\alpha_Q(j)}), dF(r, [Qr]_{\alpha_Q(r)}) \right\} \right)$$

where $\chi \in \Omega$. Then, the function Q has at least a fuzzy fixed point.

Corollary 3.12. Let (J, dF) be an \mathcal{F} -complete \mathcal{F} -metric space and $I, Q : J \rightarrow \mathcal{D}(J)$ be multi-valued functions. Suppose that for all $j, r \in J$, the following holds:

$$H_F(Ij, Qr) \leq \chi \left(\max \left\{ dF(j, r), dF(j, Ij), dF(r, Qr), \frac{dF(j, Ij) + dF(r, Qr)}{2} \right\} \right)$$

where $\chi \in \Omega$. Then, the functions I and Q have at least a common fixed point.

Corollary 3.13. Let (J, dF) be an \mathcal{F} -complete \mathcal{F} -metric space and $I, Q : J \rightarrow \mathcal{D}(J)$ be multi-valued functions. Suppose that for all $j, r \in J$, the following holds:

$$H_F(Ij, Qr) \leq \chi(\max\{dF(j, r), dF(j, Ij), dF(r, Qr)\})$$

where $\chi \in \Omega$. Then, the functions I and Q have at least a common fixed point.

Corollary 3.14. Let (J, dF) be an \mathcal{F} -complete \mathcal{F} -metric space and $Q : J \rightarrow \mathcal{D}(J)$ be a multi-valued function. Suppose that for all $j, r \in J$, the following holds:

$$H_F(Qj, Qr) \leq \chi \left(\max \left\{ dF(j, r), dF(j, Qj), dF(r, Qr), \frac{dF(j, Qj) + dF(r, Qr)}{2} \right\} \right)$$

where $\chi \in \Omega$. Then, the function Q has at least a fixed point.

Corollary 3.15. Let (J, dF) be an \mathcal{F} -complete \mathcal{F} -metric space and $Q : J \rightarrow \mathcal{D}(J)$ be a multi-valued function. Suppose that for all $j, r \in J$, the following holds:

$$H_F(Qj, Qr) \leq \chi(\max\{dF(j, r), dF(j, Qj), dF(r, Qr)\})$$

where $\chi \in \Omega$. Then, the function Q has at least a fixed point.

Corollary 3.16. Let (J, dF) be an \mathcal{F} -complete \mathcal{F} -metric space and $Q : J \rightarrow \mathcal{D}(J)$ be a multi-valued function. Suppose that for all $j, r \in J$, the following holds:

$$H_F(Qj, Qr) \leq \lambda dF(j, r)$$

where $\lambda \in (0, 1)$. Then, the function Q has at least a fixed point.

Corollary 3.17. (Jleli & Samet, 2018). Let (J, dF) be an \mathcal{F} -complete \mathcal{F} -metric space and $Q : J \rightarrow \mathcal{D}(J)$ be a single-valued function. Suppose that for all $j, r \in J$, the following holds:

$$dF(Qj, Qr) \leq \lambda dF(j, r)$$

where $\lambda \in (0, 1)$. Then, the function Q has a unique fixed point.

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