

On The Existence and Uniqueness Conditions of The MLEs of The Four-Parameter Exponentiated BurrXII-Poisson Distribution

¹Ibrahim Abdullahi, and ^{*2}Shamsuddeen Ahmad Sabo

¹Department of Mathematics, Yusuf Maitama Sule University, Kano, Nigeria.

²Department of Statistics, Aliko Dangote University of Science and Technology, Wudil, Nigeria

***Corresponding Author's Email:** shamsuddeenahmadsabo@gmail.com

Received on: August, 2024

Revised and Accepted on: September, 2024

Published on: October, 2024

ABSTRACT

A maximum likelihood estimator is guaranteed to exist, provided specific regularity conditions are met to ensure a well-defined likelihood function and achievable maximization. This paper considered the existence and uniqueness of the maximum likelihood estimates (MLE) of the exponentiated BurrXII Poisson (EBXIIP) distribution under some possible conditions.

Keywords: EBXIIP distribution; Regularity conditions; Statistical inference; MLE method; Density properties

1.0 INTRODUCTION

Maximum Likelihood Estimation (MLE) stands as a crucial statistical method widely employed for parameter estimation in various models (Osi *et al.*, 2024). It holds significance in providing estimates that maximize the likelihood function, offering efficient and asymptotically unbiased parameter values. MLE plays a pivotal role in both frequentist and Bayesian statistics, providing a solid foundation for statistical inference.

Likewise, Profile likelihood, on the other hand, is an essential tool that refines MLE by examining the behaviors of the likelihood function along a profile or restricted parameter space. It aids in constructing confidence intervals and understanding parameter uncertainties, providing a nuanced perspective on the reliability of MLE estimates.

Together, MLE and profile log likelihood enhance the precision of parameter estimates, contributing to robust statistical analyzes and informed decision-making in a wide array of disciplines.

The existence of maximum likelihood estimates is guaranteed under regularity conditions, ensuring the likelihood function is well-defined and maximization is achievable. Uniqueness is contingent on the uniqueness of the global maximum, which is often assured when the parameter space is well-behaved and the likelihood function is concave. This has been considered for many distributions by many authors in literature. For instance, the work of (Adamidis and Loukas, 1998) delves into the existence and uniqueness of MLEs of the exponential geometric. Similarly, (Kus, 2007), address this for the exponential Poisson (EP) distribution, Abdullahi *et al.* (2023), for the Transform Transmuted Rayleigh distribution, and Wang *et al.* (2022) for the enhanced

version of Weibull distribution named improved Weibull-Weibull distribution, among others. It has also been considered by Muhammad and Liu (2019) on a new extension of generalized half logistic distribution and recently by Alomair *et al.* (2024) for the exponentiated XLindley distribution. Readers are also suggested to see Zhu and Balakrishnan (2022), Shrahili *et al.* (2022), Alanzi *et al.* (2023), Muhammad *et al.* (2024) and Mohammad *et al.* (2024) for more related discussions.

It is important to note that, the existence of MLEs is not guaranteed in all cases. Tumlinson (2015) demonstrated, for example, the non-existence of MLEs for the extended exponential power distribution (EEP) and its generalizations.

Consequently, in this paper, we aim to study the shape properties of the EBXIIP. In this respect, the behaviour of the distribution's PDF is discussed. Furthermore, we establish the uniqueness and existence of the MLEs of the EBXIIP parameters.

We derive some shape properties of EBXIIP in Section 2, and discuss the existence and uniqueness of the MLEs of the exponentiated BurrXII Poisson (EBXIIP) distribution under some possible conditions in Section 3; then conclusion in Section 4.

2.0 ON THE FOUR PARAMETER EBXIIP

Here, we provided some shape and limiting properties of the EBXIIP which was not considered by (da Silva *et al.* 2015).

The EBXIIP distribution was introduced and studied by (da Silva *et al.* 2015), some important properties, estimation by maximum likelihood and applications of the EBXIIP were considered but the asymptotic behavior and the nature of roots of the maximum likelihood

estimators were not considered. A random variable X has EBXIIP if its cumulative distribution function (CDF) and probability density function (PDF) are given respectively as:

$$F(x) = \frac{1 - \exp(-\lambda(1 - (1 + x^\alpha)^{-\beta})^a)}{1 - \exp(-\lambda)} \quad (1)$$

$$f(x) = \frac{a\alpha\beta\lambda x^{\alpha-1} (1 - (1 + x^\alpha)^{-\beta})^{a-1} \exp(-\lambda(1 - (1 + x^\alpha)^{-\beta})^a)}{(1 - \exp(-\lambda))(1 + x^\alpha)^{\beta+1}} \quad (2)$$

Theorem 1: The behavior of the limit of the PDF of the EBXIIP as specified in equation (2) are:

$$(1) \lim_{x \rightarrow 0} f(x) = \begin{cases} \infty & \alpha < 1, a \leq 1 \\ 0 & \alpha > 1, a \geq 1 \end{cases}$$

$$(2) \text{ For } \alpha = 1, \lim_{x \rightarrow 0} f(x) = \begin{cases} \infty & 0 < a < 1 \\ 0 & a > 1 \\ \frac{\beta\lambda}{1 - e^{-\lambda}} & a = 1 \end{cases}$$

$$(3) \lim_{x \rightarrow \infty} f(x) = 0, \text{ for } \alpha \leq 1 \text{ and } \forall a > 0$$

(4) The limiting distribution of the (1) as $\lambda \rightarrow 0^+$ is $\lim_{\lambda \rightarrow 0^+} F(x) = (1 - (1 + x^\alpha)^{-\beta})^a$, i.e. it converges to exponentiated BurrXII distribution.

Proof: The proof is straightforward calculation by taking the limits.

Theorem 2: For $\alpha \in (0, 1]$ and $a \in (0, 1]$, the PDF of the EBXIIP as expressed by (2) is a monotone decreasing function, and unimodal for $\alpha > 1$ and $a > 1$.

Proof: The first derivative of $\log f(x)$ in (2) is:

$$\frac{d(\log f(x))}{dx} = \frac{\alpha - 1}{x} - \frac{\alpha(\beta + 1)x^{\alpha-1}}{(1 + x^\alpha)} + \frac{(a - 1)\alpha\beta x^{\alpha-1}}{(1 + x^\alpha)^{\beta+1}(1 - (1 + x^\alpha)^{-\beta})} - \frac{a\alpha\beta\lambda x^{\alpha-1}(1 - (1 + x^\alpha)^{-\beta})^{a-1}}{(1 + x^\alpha)^{\beta+1}}$$

For $\alpha \in (0, 1]$ and $a \in (0, 1]$, $d(\log f(x))/dx < 0$, thus, $f(x)$ is monotone decreasing. Suppose that $a > 1$, then $f(x)$ has exactly one root say x_0 . Therefore, if; $x < x_0$, then $f(x) > 0$, and if $x > x_0$, then $f(x) < 0$, hence, $f(x)$ is a unimodal with mode at $x = x_0$.

3.0 MAIN RESULTS

Here, we provided the maximum likelihood estimation (MLE) equations as given in (da Silva *et al.* 2015). Consider $X_i (i = 1, 2, \dots, n)$ as a randomly selected sample of size n obtained from the EBXIIP model where the observed values are x_1, x_2, \dots, x_n . The MLEs of $\theta = (\alpha, \beta, \lambda, \zeta)^T$ are determined by maximizing the function \log - likelihood of θ ($\log \ell(\theta)$) expressed by

$$\log \ell(\theta) = n \log a + n \log \alpha + n \log \beta + n \log \lambda - n \log(1 - \exp(-\lambda))$$

$$+ (\alpha - 1) \sum_{i=1}^n \log x_i - (\beta + 1) \sum_{i=1}^n \log(1 + x_i^\alpha)$$

$$+ (\alpha - 1) \sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta}) - \lambda \sum_{i=1}^n (1 - (1 + x_i^\alpha)^{-\beta})^a$$

Alternatively, through the concurrent resolution of the subsequent nonlinear equations:

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log x_i - (\beta + 1) \sum_{i=1}^n \frac{x_i^\alpha \log x_i}{(1 + x_i^\alpha)} + (a - 1) \beta \sum_{i=1}^n \frac{x_i^\alpha \log x_i}{(1 - (1 + x_i^\alpha)^{-\beta})(1 + x_i^\alpha)^{\beta+1}}$$

$$- \alpha \beta \lambda \sum_{i=1}^n \frac{x_i^\alpha \log x_i (1 - (1 + x_i^\alpha)^{-\beta})^{a-1}}{(1 + x_i^\alpha)^{\beta+1}} \quad (3)$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^n \log(1 + x_i^\alpha) - a \lambda \sum_{i=1}^n \frac{\log(1 + x_i^\alpha) (1 - (1 + x_i^\alpha)^{-\beta})^{a-1}}{(1 + x_i^\alpha)^\beta}$$

$$+ (a - 1) \sum_{i=1}^n \frac{\log(1 + x_i^\alpha)}{(1 - (1 + x_i^\alpha)^{-\beta})(1 + x_i^\alpha)^\beta} \quad (4)$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \frac{n \exp(-\lambda)}{(1 - \exp(-\lambda))} - \sum_{i=1}^n (1 - (1 + x_i^\alpha)^{-\beta})^a \quad (5)$$

$$\frac{\partial \ell}{\partial a} = \frac{n}{a} + \sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta}) - \lambda \sum_{i=1}^n ((1 - (1 + x_i^\alpha)^{-\beta})^a) \log(1 - (1 + x_i^\alpha)^{-\beta}) \quad (6)$$

Now, we can discuss the uniqueness and existence of the MLEs of the parameters α , β and λ under certain potential conditions. Similarly, the properties, uniqueness and existence for the complementary exponentiated BurrXII-Poisson (CEBXIIP) MLEs were examined by (Mustapha, 2017).

Theorem 3: Suppose $g_1(\alpha, a, \beta, \lambda, x_i)$ represent the function on the RHS of (3), where α , β and λ are the actual parameter values, then, the root of $g_1(\alpha, a, \beta, \lambda, x_i) = 0$ can take one of the following conditions.

- (a) For $Max\{X_i\} < 1$ and $a \geq 1$
- (b) For $Min\{X_i\} > 1$ and $a \geq 1$

Proof: for $a \geq 1$, $\lim_{\alpha \rightarrow 0} g_1(\alpha, \beta, \lambda, x_i) = \infty$ & $\lim_{\alpha \rightarrow \infty} g_1(\alpha, \beta, \lambda, x_i) = a \sum_{x_i < 1}^n \log x_i - \beta \sum_{x_i > 1}^n \log x_i$. To show that $g_1 < 0$ as $\alpha \rightarrow \infty$, we consider these cases.

1. If $Max\{X_i\} < 1$, then, $\lim_{\alpha \rightarrow \infty} g_1(\alpha; \beta, \lambda, x_i) = a \sum_{x_i < 1}^n \log x_i < 0$
2. If $Max\{X_i\} > 1$, then, $\lim_{\alpha \rightarrow \infty} g_1(\alpha; \beta, \lambda, x_i) = -\beta \sum_{x_i > 1}^n \log x_i < 0$
3. If $Min\{X_i\} > 1$ and $Max\{X_i\} < 1$, then,

$$\lim_{\alpha \rightarrow \infty} g_1(\alpha; \beta, \lambda, x_i) = a \sum_{x_i < 1}^n \log x_i - \beta \sum_{x_i > 1}^n \log x_i < 0$$

Therefore, $g_1 < 0$ in all cases, if and only if $x \neq 1$, for some $i = 1, 2, \dots, n$. Since g_1 is continuous, thus, there exist at least one root for $g_1(\alpha; \beta, \lambda, x_i) = 0$.

Theorem 4: Suppose $g_2(\beta; a, \alpha, \lambda, x_i)$ represent the function on the RHS of (4), where a, α and λ are the actual parameter values, then, the equation $g_2(\beta; a, \alpha, \lambda, x_i) = 0$, has at least a root for $a \neq 1$ and for $a = 1$, the root lies in the interval $(n((1 + \lambda) \sum_{i=1}^n \log(1 + x_i^\alpha))^{-1}, n(\sum_{i=1}^n \log(1 + x_i^\alpha))^{-1})$.

Proof: For $a = 1$, let $w_2 = -\lambda \sum_{i=1}^n \frac{\log(1 + x_i^\alpha)}{(1 + x_i^\alpha)^\beta}$, w_2 is strictly decreasing in β , then,

$\lim_{\beta \rightarrow 0} w_2 = -\lambda \sum_{i=1}^n \log(1 + x_i^\alpha)$ and $\lim_{\beta \rightarrow \infty} w_2 = 0$, thus,

$$g_2(\beta; a, \alpha, \lambda, x_i) > \frac{n}{\beta} - \sum_{i=1}^n \log(1 + x_i^\alpha) + \lim_{\beta \rightarrow 0} w_2 = \frac{n}{\beta} - (\lambda + 1) \sum_{i=1}^n \log(1 + x_i^\alpha),$$

Then,

$$g_2(\beta; a, \alpha, \lambda, x_i) > 0 \text{ When } \beta < \frac{n}{(\lambda + 1) \sum_{i=1}^n \log(1 + x_i^\alpha)}$$

and

$$g_2(\beta; a, \alpha, \lambda, x_i) < \frac{n}{\beta} - \sum_{i=1}^n \log(1 + x_i^\alpha) + \lim_{\beta \rightarrow \infty} w_2 = \frac{n}{\beta} - \sum_{i=1}^n \log(1 + x_i^\alpha)$$

Then,

$$g_2(\beta; a, \alpha, \lambda, x_i) < 0 \text{ when } \beta > \frac{n}{\sum_{i=1}^n \log(1 + x_i^\alpha)}$$

Hence, $g_2 = 0$ possesses minimum of one root in $(n(1 + \lambda) \sum_{i=1}^n \log(1 + x_i^\alpha))^{-1}, n(\sum_{i=1}^n \log(1 + x_i^\alpha))^{-1})$

For $a \neq 1$

$\lim_{\beta \rightarrow 0} g_2 = \infty$, then, we prove that $\lim_{\beta \rightarrow \infty} g_2 < 0$.

$\lim_{\beta \rightarrow \infty} g_2 = -\sum_{i=1}^n \log(1 + x_i^\alpha) < 0$, since g_2 is continuous function and monotone which declines from positive to negative values, therefore, $g_2(\beta; a, \alpha, \lambda, x_i) = 0$ has at least a root.

Theorem 5: Suppose $g_3(\lambda; a, \alpha, \beta, x_i)$ represent the function on the RHS of (5); a, α and β be the actual parameter values, then, $g_3(\lambda; a, \alpha, \beta, x_i) = 0$ gets at minimum of one root for $\frac{n}{2} > \sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta})^a$.

Proof: Let $w_3 = n \left[\frac{1}{\lambda} - \frac{\exp(-\lambda)}{1 - \exp(-\lambda)} \right]$, w_3 is strictly decreasing in λ , then, $\lim_{\lambda \rightarrow \infty} w_3 = 0$, therefore,

$$g_3 < \lim_{\lambda \rightarrow 0} w_3 - \sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta})^a = -\sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta})^a < 0.$$

Then again, $\lim_{\lambda \rightarrow 0} w_3 = \frac{n}{2}$, therefore,

$$g_3 > \lim_{\lambda \rightarrow 0} w_3 - \sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta})^a = \frac{n}{2} - \sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta})^a < 0, \text{ thus,}$$

$$g_3 > 0 \text{ when } \frac{n}{2} > \sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta})^a.$$

Hence, $g_3 = 0$ has at least one root when $\frac{n}{2} > \sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta})^a$

Theorem 6: Let $g_4(a; \alpha, \beta, \lambda, x_i)$ be the function on the RHS of (6), where α, β and λ are the actual parameter values, then for $\lambda < 1$ the root of $g_4(a; \alpha, \beta, \lambda, x_i) = 0$ lies in the interval

$$\left(\frac{-n}{\sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta})}, \frac{-n}{(1 - \lambda) \sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta})} \right), \text{ and is unique.}$$

Proof: Let $w_4 = -\lambda \sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta})^a \log(1 - (1 + x_i^\alpha)^{-\beta})$, w_4 is strictly decreasing in a , and $\lim_{a \rightarrow 0} w_4 = -\lambda \sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta})$, we have that,

$$g_4 < \frac{n}{a} + \sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta}) + \lim_{a \rightarrow 0} w_4 = \frac{n}{a} - (\lambda - 1) \sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta}), \text{ thus, } g_4 < 0 \text{ if}$$

$$a < \frac{-n}{(1 - \lambda) \sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta})}.$$

On one hand, $\lim_{a \rightarrow \infty} w_4 = 0$, so,

$$g_4 > \frac{n}{a} + \sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta}) + \lim_{a \rightarrow \infty} w_4 = \frac{n}{a} + \sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta}), \text{ thus, } g_4 > 0 \text{ whenever}$$

$a > \frac{-n}{\sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta})}$, hence we have that, for $\lambda < 1$ the root of $g_4 = 0$, lies in

$$\left(\frac{-n}{\sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta})}, \frac{-n}{(1 - \lambda) \sum_{i=1}^n \log(1 - (1 + x_i^\alpha)^{-\beta})} \right). \text{ To demonstrate the uniqueness of the root, we}$$

show that $g_4' < 0$. And $g_4' = -\frac{n}{a^2} - \lambda \sum_{i=1}^n (1 - (1 + x_i^\alpha)^{-\beta})^a (\log(1 - (1 + x_i^\alpha)^{-\beta}))^2 < 0$.

4.0 CONCLUSION

In this work, the maximum likelihood estimate's existence and uniqueness for the EBXIIIP are investigated under several possible settings. Additionally, some shape properties of the density of the EBXIIIP are also derived.

ACKNOWLEDGEMENTS

The authors express their gratitude to the editor and the anonymous referees for their insightful comments which significantly enhanced the quality of this manuscript.

REFERENCES

- Abdullahi, J., Gulumbe, S.U., Usman, U., Garba, A.I. (2023). The Transform-Transformer Approach: Unveiling the odd transmuted Rayleigh-X Family of Distributions. *International journal of science for global sustainability*, 9(2): 85-98. DOI: <https://doi.org/10.57233/ijsgs.v9i2.462>
- Adamidis, K., & Loukas, S. (1998). A lifetime distribution with decreasing failure rate. *Statistics and Probability Letters*, 39(1), 35-42. DOI: [https://doi.org/10.1016/s0167-7152\(98\)00012-1](https://doi.org/10.1016/s0167-7152(98)00012-1)

- Alanzi R.A., Imran, M., Tahir, M.H., Chesneau, C., Jamal, F., Shakoor, S., & Sami, W. (2023). Simulation analysis, properties on a new Burr XII model based on the Bell-X functionalities. *Aims Mathematics*, 8(3), 6970-7004. DOI: <http://dx.doi.org/10.3934/math.2023352>.
- Alomair, A. M., Ahmed, M., Tariq, S., Ahsan-ul-Haq, M., & Talib, J. (2024). An exponentiated XLindley distribution with properties, inference and applications. *Heliyon*, 10(3). DOI: <http://dx.doi.org/10.1016/j.heliyon.2024.e25472>
- da Silva, R.V., Gomes-Silva, F., Ramos, M.W.A, & Cordeiro, G.M. (2015). The exponentiated Burr XII Poisson distribution with application to life data. *International Journal of Statistics and Probability*, 4(4), 112-131. DOI: <http://dx.doi.org/10.5539/ijsp.v4n4p112>
- Kus C. (2007). A new lifetime distribution. *Computational Statistics & Data Analysis*. 51(9), 4497-4509. DOI: <https://doi.org/10.1016/j.csda.2006.07.017>



- Mahmoudi, E., & Jafari, A.A. (2012). Generalized exponential-power series distributions. *Computational Statistics and Data Analysis*, 56(12), 4047-4066. DOI: <https://doi.org/10.1016/j.csda.2012.04.009>
- Mohammad, A.S., Silas, N., Abdullahi, I., Abdullahi, J., Abba, B. (2024). Classical and Bayesian inferences of a new additive Weibull extension and distribution with application. *International journal of science for global sustainability*, 10(2):75-85. DOI: <https://doi.org/10.57233/ijsgs.v10i2.650>
- Muhammad, M., & Liu, L. (2019). A new extension of the generalized half logistic distribution with applications to real data. *Entropy*, 21(4), 339. DOI: <http://dx.doi.org/10.3390/e21040339>
- Muhammad, Mustapha, Isyaku Muhammad, Mouna Bouchane, Muhammad Aslam, Sani Musa, and Sadiya Ali Rano. (2024). A New Three Parameter Bathtub and Increasing Failure Rate Model with Applications to Real Data. *Pakistan Journal of Statistics and Operation Research* 20(1), 49–67. DOI: <http://dx.doi.org/10.18187/pjsor.v20i1.3614>
- Mustapha, M. (2017). The complementary Exponentiated BurrXII Poisson Distribution: model, properties and application. *Journal of Statistics Applications & Probability*, 6(1), 33-48. DOI: <http://dx.doi.org/10.18576/jsap/060104>
- Osi, A.A., Ahmad, S.S., Ibrahim, Z.M. (2024). Inverted Dagum Distribution: properties and application. *RT&A*, 3(19), 595-604. DOI: <https://doi.org/10.24412/1932-2321-2024-379-595-604>
- Shrahili, M., Mustapha Muhammad, Ibrahim Elbatal, Isyaku Muhammad, Mouna Bouchane, and Badamasi Abba. (2022). Properties and Applications of the Type I Half-Logistic Nadarajah-Haghighi Distribution. *Austrian Journal of Statistics*. DOI: <http://dx.doi.org/10.17713/ajs.v52i2.1363>
- Tumlinson, S.E. (2015). On the non-existence of maximum likelihood estimates for the extended exponential power distribution and its generalizations. *Statistics and Probability Letters*, 107, 111-114. DOI: <https://doi.org/10.1016/j.spl.2015.08.009>
- Wang, Hong, Badamasi Abba, and Jianxin Pan. (2022). Classical and Bayesian Estimations of Improved Weibull-Weibull Distribution for Complete and Censored Failure Times Data. *Applied Stochastic Models in Business and Industry* 38(6), 997–1018. DOI: <http://dx.doi.org/10.1002/asmb.2698>
- Zhu, Xiaojun, and N. Balakrishnan. (2022). One-Shot Device Test Data Analysis Using Non-Parametric and Semi-Parametric Inferential Methods and Applications. *Reliability Engineering and System Safety* 221(June 2021), 108319. DOI: <http://dx.doi.org/10.1016/j.ress.2022.108319>