

Bi-dual and Reflexivity of Cone Normed Space

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ABSTRACT

A cone normed space is one of the generalizations of normed space in which the codomain of normed space is replaced by an ordered real Banach space. In this paper, some properties of the dual, in particular bi-dual in the setting of cone normed space were studied. This allows us to define the notion of reflexivity with respect to cone normed space and proved the necessary and sufficient condition for a cone normed space to be reflexive.

Keywords: Dual, Bi-dual, Reflexivity, Cone normed space, Weak* topology

1.0 INTRODUCTION

Rzepecki (1980) was the first to introduce the notion of cone metric and cone norm where the metric is defined in such way that $d_E: X \times X \rightarrow S$ where d_E is a metric on a set X , E is a Banach space and S is a normal cone in E with a partial order " \leq ". This concept serves as a generalization of a metric on a set X known as the cone metric on X . Later on (Huang & Zhang, 2007) gives another definition of cone metric where they replaced the space of real numbers as the codomain of metric by an ordered real Banach space and proved some fixed point theorems of contractive mapping in this new setting. Since then, many authors have used this concept in generalizing the notion of cone metric space to cone normed space. As an example, (Eshaghi *et al.*, 2012) proved Banach Steinhaus theorem and Baire's category in cone normed space while (Tamang & Bag, 2019) studied some results in fixed point of fuzzy cone normed linear space where they extend the fixed point theorems of Banach, Kannan and Chatterjee to fuzzy cone normed linear space. Also, (Yusuf *et al.*, 2020) studied the dual of cone normed space and proved the Hahn-Banach theorem in the setting of cone normed spaces. Donuan & Jonard-Perez (2020) proved that the topological structure and the covering dimension of an asymmetric normed space with finite dimension may be expressed in terms of specific algebraic properties. A study on nonsymmetric cone normed space was done by (Ilkhan, 2020) where they introduced some topological concepts of asymmetric cone normed spaces. Yang & Zhu (2021) gives some concepts of cone b-norm and cone b-Banach spaces and established some fixed point theorems. I-convergence in fuzzy cone normed space was studied by (Guler, 2021) while (Yusuf & Garba, 2021) extend the concept of cone normed space to cone inner product space over Banach algebra and establish some properties such as Cauchy Schwarz inequality, parallelogram identity and Pythagoras theorem. Various types of cone metric space and their characterization was given and studied by (Abou Bakr & Mohamed, 2023) and also considered

some generalized contraction types of mapping on θ -complete cone metric like spaces. Gurdal (2023) extended some existing theories on convergence sequences in fuzzy cone normed space to lacunary ideal convergence. The purpose of this paper therefore is to investigate some properties of dual and bi-dual of cone normed space and establish the necessary and sufficient condition for a cone normed space to be reflexive. A non-empty subset P of a Banach space E is said to be a cone if

- i) $P + P \subseteq P$ Closure under vector addition,
- j) $\lambda P \subseteq P \forall \lambda \geq 0$ Closure under scalar multiplication.

If in addition, $P \cap \{-P\} = \{0\}$, then P is called a pointed cone. Some authors, used to refer to a cone as a pointed cone as simply a cone and a cone as a wedge. We define a partial ordering \leq with respect to P by $x \leq y$ if and only if $y - x \in P$ while the ordering " \ll " with respect to P by $x \ll y$ if and only if $y - x \in \text{int } P$, for a given cone $P \subseteq E$, where E is a Banach space, $\text{int } P$ denote the interior of P . Note that $y - x \in \text{int } P$ implies $y - x \in P$ but the reverse is not always the case. Thus, $x \ll y \Rightarrow x \leq y$.

2.0 PRELIMINARY

Definition 1 (Eshaghi *et al.*, 2012): Let X be a real vector space. Suppose that a mapping $\|\cdot\|_c: X \rightarrow E$ where E is a real Banach space, satisfies:

- 1) $\|x\|_c \geq 0 \forall x \in X$,
- 2) $\|x\|_c = 0 \Leftrightarrow x = 0 \forall x \in X$,
- 3) $\|\alpha x\|_c = |\alpha| \|x\|_c \forall x \in X$ and $\alpha \in \mathbb{R}$,
- 4) $\|x + y\|_c \leq \|x\|_c + \|y\|_c \forall x, y \in X$.

Then, $\|\cdot\|_c$ is a cone norm on X and the pair $(X, \|\cdot\|_c)$ a cone normed space.

For example of cone normed spaces, see (Eshaghi *et al.*, 2012).

Definition 2 (Eshaghi *et al.*, 2012): Let $(X, \|\cdot\|_c)$ be a cone normed space. A sequence $\{x_n\} \in X$ is said to be



- 1) Cauchy, if for every $0 \ll c$ with $c \in E$ there exist $N \in \mathbb{N}$ such that $\|x_n - x_m\|_c \ll c \forall n, m \geq N$.
- 2) convergent to x , if for every $0 \ll c$ with $c \in E$ there exist $N \in \mathbb{N}$ such that $\|x_n - x\|_c \ll c \forall n \geq N$

$(X, \|\cdot\|_c)$ is said to be a complete cone normed space or cone Banach space if every Cauchy sequence converge.

Throughout this paper, E will be a real Banach space, P a cone, $\|\cdot\|_c$ a cone norm, $X_c = (X, \|\cdot\|_c)$ a cone normed space and $(X, \|\cdot\|)$ the classical normed space. $\mathcal{L}(X_c, Y_c)$ will be the linear space of all continuous linear mappings from X_c to Y_c , i.e $T: X_c \rightarrow Y_c$ such that $T \in \mathcal{L}(X_c, Y_c)$, and $\mathcal{L}(X, Y)$ the space of all continuous linear mappings from the classical normed space $(X, \|\cdot\|)$ to $(Y, \|\cdot\|)$.

Lemma 1 (Eshaghi *et al.*, 2012): Let X_c and Y_c be two cone normed spaces and $f: X_c \rightarrow Y_c$ be a linear mapping. Then the following are equivalent:

- i) (Continuity at a point) For some fixed $x_0 \in X$ we have: Given $0 \ll c$ there is a $0 \ll t$ such that $\|f(x) - f(x_0)\|_c \ll c$ whenever $\|x - x_0\|_c \ll t$
- ii) (Continuity at zero) For $0 \ll c$ there is a $0 \ll t$ such that $\|f(x)\|_c \ll c$ whenever $x \in X$ and $\|x\|_c \ll t$
- iii) (Continuity at every point) For any $x \in X$ we have: Given $0 \ll c$ there is a $0 \ll t$ such that $\|f(x) - f(y)\|_c \ll c$ whenever $y \in X$ and $\|y - x\|_c \ll t$
- iv) (Uniform continuity) given $0 \ll c$ there is a $0 \ll t$ such that $\|f(x) - f(y)\|_c \ll c$ whenever $x, y \in X$ and $\|x - y\|_c \ll t$
- v) (Sequential continuity) Given any sequence $\{x_n\} \subseteq X$ which is convergent to a point $x_0 \in X$, the sequence $\{f(x_n)\} \subseteq Y$ is convergent to a point $f(x_0) \in Y$

Definition 3 (Yusuf *et al.*, 2020): (Cone bounded map) Let X_c and Y_c be two cone normed spaces and $T: X_c \rightarrow Y_c$ be a linear mapping. Then, T is said to be cone bounded if there exist an $M \geq 0$ such that $\|T(x)\|_c \leq M \|x\|_c$ for all $x \in X_c$.

Lemma 2 (Yusuf *et al.*, 2020): Let X_c and Y_c be two cone normed spaces and $T: X_c \rightarrow Y_c$ be a linear mapping. If T is cone bounded, then $T \in \mathcal{L}(X_c, Y_c)$.

Proof: Assume that T is cone bounded, then there exist $M \geq 0$ such that $\|T(x)\|_c \leq M \|x\|_c$ for all $x \in X$. Let $0 \ll c$ be given and consider an arbitrary $x_0 \in X$, then we can choose $t = \frac{c}{M} \gg 0$ such that $\|x - x_0\|_c \ll t$ for all $x \in X$. Now, for any $x \in X$

$$\begin{aligned} \|T(x) - T(x_0)\|_c &= \|T(x - x_0)\|_c \text{ since } T \text{ is linear} \\ &\leq M \|x - x_0\|_c \text{ since } T \text{ is cone bounded} \\ &\ll M \times t = c \end{aligned}$$

Since x_0 was arbitrary, T is continuous.

3.0 Bidual OF CONE NORMED SPACE

The dual of a cone normed space was defined by (Yusuf *et al.*, 2020) in such away that

$$\begin{aligned} X_c^* &= \{f: (X, \|\cdot\|_c) \\ &\rightarrow (\mathbb{R}, |\cdot|)\} \text{ where } f \text{ is linear and continuous} \end{aligned}$$

Since \mathbb{R} is complete with respect to $\|\cdot\|_c = |\cdot|$, X_c^* is a cone Banach space.

We begin this section by stating some important properties of the dual of cone normed space.

Definition 4: Let $X_c = (X, \|\cdot\|_c)$ be a cone normed space, we denote by $B_{X_c}, B_{X_c^*}$ the unit ball of both the cone normed space and the dual cone normed space X_c^* such that:

$$B_{X_c} = \{x \in X_c: \|x\|_c \ll 1, \text{ with } 1 \in \text{int}P\}$$

$$B_{X_c^*} = \{f \in X_c^*: \|f\|_c \ll 1, \text{ with } 1 \in \text{int}P\}$$

respectively.

Proposition 1: Let $X_c = (X, \|\cdot\|_c), X = (X, \|\cdot\|)$ be a cone normed space and normed space respectively. Then,

$$B_{X_c^*} \subseteq B_{X^*}$$

Proof: We need to show that

$$i) f \in B_{X_c^*} \Rightarrow f \in X_c^*$$

$$ii) f \in X_c^* \Rightarrow f \in X^* \text{ and}$$

$$iii) \|f\|_c \ll 1 \Rightarrow \|f\| \leq 1$$

For i) we know by definition that $B_{X_c^*} = \{f \in X_c^*: \|f\|_c \ll 1, \text{ with } 1 \in \text{int}P\}$, thus $f \in B_{X_c^*} \Rightarrow f \in X_c^*$.

For ii) $f \in X_c^* \Rightarrow f \in X^*$ because $X_c^* = \{f: (X, \|\cdot\|_c) \rightarrow (\mathbb{R}, |\cdot|)\}$ where f is linear and continuous } which is also the case in X^* .

And For iii) since $\|f\|_c = |f| \ll 1$ and $\|f\| = |f|$ we have that $\|f\| \ll 1$ but $\|f\| \ll 1$ implies that $\|f\| \leq 1$.

For our next result, we first state the following theorem which will be useful in the proof.

Theorem 1 (Yusuf *et al.*, 2020): (Hahn-Banach Theorem for Cone normed space) Let f be a cone bounded linear functional on a subspace M of a cone normed space X_c . Then there exist a cone bounded linear functional F on X_c which is an extension of f to X_c and has the same cone norm $\|F\|_c = \|f\|_c$.

Lemma 3: Let X_c be a cone normed space, for each $x_0 \in X \setminus \{0\}$ there exist $f_0 \in X_c^*$ such that $f_0(x_0) = \|x_0\|_c$ and $\|f_0\|_c = 1$.

Proof: Let M be a subspace of X_c consisting of all elements $x = \alpha x_0$ where $\alpha \in \mathbb{R}$. Define a linear functional f on M by $f(x) = f(\alpha x_0) = \alpha \|x_0\|_c$. Then f is cone bounded with cone norm $\|f\|_c = 1$, because

$$|f(x)| = |f(\alpha x_0)| = |\alpha| \|x_0\|_c = \|\alpha x_0\|_c = \|x\|_c$$

Hence, By Theorem (1), f has a linear extension f_0 from M to X_c such that $\|f_0\|_c = \|f\|_c = 1$. Hence $f_0(x_0) = f(x_0) = \|x_0\|_c$.

Theorem 2: Let $X_c = (X, \|\cdot\|_c)$ be a cone normed space. Then for each $x \in X_c, \|x\|_c = \sup\{|f(x)| : f \in X_c^*, \|f\|_c = 1\}$.

Proof: If $x = 0$, then the result follows immediately. Assume that $x \in X \setminus \{0\}$. For any $f \in X_c^*$ with $\|f\|_c = 1$,

$$|f(x)| \leq \|f\|_c \|x\|_c = \|x\|_c$$

Hence, $\sup\{|f(x)| : f \in X_c^*, \|f\|_c = 1\} \leq \|x\|_c$.

On the other hand, by Lemma (3) there exists $f \in X_c^*$ such that $\|f\|_c = 1$ and $f(x) = \|x\|_c$. Therefore, $\|x\|_c = |f(x)| \leq \sup\{|f(x)| : f \in X_c^*, \|f\|_c = 1\}$. Hence, $\|x\|_c = \sup\{|f(x)| : f \in X_c^*, \|f\|_c = 1\}$.

Let $X_c = (X, \|\cdot\|_c)$ be a cone normed space. By analogy with the notion of the dual of a cone normed space X_c^* introduced by (Yusuf *et al.*, 2020), we define the set:

$$X_c^{**} = \{\psi : (X_c^*, \|\cdot\|_c) \rightarrow (\mathbb{R}, \|\cdot\|_c) \mid \psi \text{ is a linear and continuous}\}$$

Now, for each $\psi \in X_c^{**}$, if we set

$$\|\psi\|_c = \sup\{|\psi(f)| : \|f\|_c \ll 1, f \in X_c^*\}$$

Then, $(X_c^{**}, \|\cdot\|_c)$ is a cone normed space, which will be called the bidual of a cone normed space $X_c = (X, \|\cdot\|_c)$.

Definition 5: Let $X_c = (X, \|\cdot\|_c)$ and $Y_c = (Y, \|\cdot\|_c)$ be two cone normed spaces. A linear mapping $f: X_c \rightarrow Y_c$, is called an isometric isomorphism from $X_c = (X, \|\cdot\|_c)$ to $Y_c = (Y, \|\cdot\|_c)$ if for all $x \in X, \|f(x)\|_c = \|x\|_c$.

Remark 1:

- i) Every isometric isomorphism is a one-to-one mapping.
- ii) Two cone normed spaces $X_c = (X, \|\cdot\|_c)$ and $Y_c = (Y, \|\cdot\|_c)$ are said to be isometrically isomorphic if there exists an isometric isomorphism from $X_c = (X, \|\cdot\|_c)$ onto $Y_c = (Y, \|\cdot\|_c)$.

Proposition 2: Let $T: (X, \|\cdot\|) \rightarrow (Y, \|\cdot\|)$ be a linear mapping which is isometrically isomorphic between the two normed spaces. If the cone norm is defined by

$$\|\cdot\|_c := \left\{ \frac{\|\cdot\|}{2^n} \right\}_1^\infty$$

Then, the linear mapping $T: X_c \rightarrow Y_c$ between the cone normed spaces X_c and Y_c is also isometrically isomorphic

Proof: Let the mapping $T: (X, \|\cdot\|) \rightarrow (Y, \|\cdot\|)$ be isometrically isomorphic. Then, $\|T(x)\| = \|x\|$. We want to show that $\|T(x)\|_c = \|x\|_c$ for all $x \in X$.

$$\begin{aligned} \|T(x)\|_c &= \left\{ \frac{\|T(x)\|}{2^n} \right\} \text{ by definition of cone norm} \\ &= \left\{ \frac{\|x\|}{2^n} \right\} \text{ since } \|T(x)\| = \|x\| \\ &= \|x\|_c \end{aligned}$$

Theorem 3: Let $X_c = (X, \|\cdot\|_c)$ be a cone normed space. For each $x \in X$, let

$$\psi_x: X_c^* \rightarrow \mathbb{R}$$

defined by

$$\psi_x(f) = f(x), f \in X_c^*$$

Let also $\psi(X) = \{\psi_x : x \in X\}$. Then the following hold:

- 1) ψ_x is a bounded linear functional and that $\|x\|_c = \|\psi_x\|_c$
- 2) $\psi(X)$ is a linear space which is algebraically closed in X_c^{**} , and $(\psi(X), \|\cdot\|_c)$ is a cone normed space.
- 3) $(\psi(X), \|\cdot\|_c)$ is isometrically isomorphic to $(X, \|\cdot\|_c)$
- 4) $(\psi(X), \|\cdot\|_c)$ is a cone Banach space if $(X, \|\cdot\|_c)$ is a cone Banach space.

Proof:

- 1) To show that ψ_x is linear, pick $x_1, x_2 \in X, f \in X_c^*$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} \psi_{(\alpha x_1 + \beta x_2)}(f) &= f(\alpha x_1 + \beta x_2) \\ &= f(\alpha x_1) + f(\beta x_2) \\ &= \alpha f(x_1) + \beta f(x_2) \\ &= \alpha \psi_{x_1}(f) + \beta \psi_{x_2}(f) \end{aligned}$$

Now let f be an arbitrary function in X_c^* and fix $x \in X$. Then

$$|\psi_x(f)| = |f(x)| \leq \|f\|_c \|x\|_c$$

Hence, f is a bounded linear functional. To show that $\|x\|_c = \|\psi_x\|_c$, we know by Theorem (2) that $\|x\|_c = \sup\{|f(x)| : f \in B_{X_c^*}\}$, but by definition $\psi_x(f) = f(x)$, hence $\|x\|_c = \|\psi_x\|_c$.

2) Clearly, $\psi(X)$ is a linear space, but since ψ_x is cone bounded by (1) above, hence by Lemma (2) it is continuous from X_c^* to \mathbb{R} , consequently $\psi_x \in X_c^{**}$. Therefore, $\psi(X)$ is algebraically closed in X_c^{**} . Thus, $(\psi(X), \|\cdot\|_c)$ is a cone normed space.

3) Let $\Psi: X \rightarrow \psi(X)$ defined by $\Psi(x) = \psi_x(f) \forall x \in X$. The mapping defined is linear since $\forall x_1, x_2 \in X$ and $\alpha, \beta \in \mathbb{R}$.

$$\begin{aligned} \Psi(\alpha x_1 + \beta x_2) &= \psi_{(\alpha x_1 + \beta x_2)}(f) \\ &= \alpha \psi_{x_1}(f) + \beta \psi_{x_2}(f) \\ &= \alpha \Psi(x_1) + \beta \Psi(x_2) \end{aligned}$$

The mapping is also an isometry since $\|\Psi(x)\|_c = \|\psi_x(f)\|_c = \|x\|_c$ by (1) above. Hence the mapping is isometrically isomorphic since $\Psi(x)$ is clearly an onto mapping.

4) Since X and $\psi(X)$ are isometrically isomorphic, if X is cone Banach space, then $\psi(X)$ will also be cone Banach space and if $\psi(X)$ is cone Banach space, then X will also be cone Banach space.



We can now define a canonical embedding of X_c into X_c^{**} .

Definition 6:(Canonical mapping) Let $X_c = (X, \|\cdot\|_c)$ be a cone normed space and X_c^{**} be its bi-dual. A map

$$J_c: X_c \rightarrow X_c^{**} \text{ defined by}$$

$$J_c x = \psi_x \text{ for } x \in X$$

is called a canonical or natural embedding of X_c into X_c^{**} .

The above map is linear and preserves the cone norm: that is

$$\|x\|_c = \|\psi_x\|_c = \|J_c x\|_c$$

Definition 7: (Reflexive Cone Normed space) A cone normed space $(X, \|\cdot\|_c)$ is said to be Reflexive, if the canonical embedding J_c defined above is surjective.

It is known that a cone metric space d on X induces a topology τ_c which has a sub-basis $\beta = \{B_e(x): x \in X, 0 \ll e\}$ for a topology on X , where $B_e(x) = \{y \in X: d(x, y) \ll e, x \in X, 0 \ll e\}$. This topology have been proved to be Hausdorff topology.

Thus, if $\|\cdot\|_c$ is a cone norm on X . Then, $\tau_{\|\cdot\|_c}$ is the topology induced by $\|\cdot\|_c$. The family of sets

$$V_c(0) := \{x \in X: \|x\|_c \ll c, 0 \ll c\}$$

forms a fundamental systems of neighborhood of zero for $\tau_{\|\cdot\|_c}$. Therefore, for all $x \in X$, the sets $V_c(x) = \{x + V_c(0), 0 \ll c\}$ defines a fundamental system of neighborhood of x .

Definition 8: Let X_c be a cone normed space and $M \subset X_c^*$. The weak topology on X_c induced by the family M , denoted by $\sigma(X_c, M)$ is the weakest topology on X_c with respect to which each $f \in M$ is continuous. While the weak* topology on X_c^* is defined as the weak topology generated by X_c and is denoted by $\sigma(X_c^*, X_c)$.

Definition 9: Let X_c be a cone normed space. The fundamental system of neighborhood in weak topology consists of the set of the form:

$$V_{c, f_1, \dots, f_n} := \{x \in X: |f_i(x)| \ll c, i = 1, \dots, n\}$$

where $f \in X_c^*, n \in \mathbb{N}$ and $0 \ll c$.

Definition 10: Let X_c be a cone normed space. The fundamental system of neighborhood in weak* topology consists of the set of the form:

$$V_{c, x_1, \dots, x_n} := \{f \in X_c^*: |f(x_i)| \ll c, i = 1, \dots, n\}$$

where $x_i \in X_c, n \in \mathbb{N}$ and $0 \ll c$.

Theorem 4: Let X_c be a cone Banach space. Then, X_c is reflexive if and only if the closed unit ball of X_c is compact in the weak topology.

Proof: " \Rightarrow " Assume that X_c is reflexive. Let $(f_\alpha)_{\alpha \in \Delta}$ be a net in $B_{X_c^*}$. But $B_{X_c^*} \subseteq B_{X_c^*}$, and by classical Alaoglu theorem $B_{X_c^*}$ is compact for the weak topology

on X^* , this implies there exist a subnet $(f_{\alpha\beta})_{\beta \in \Lambda}$ of $(f_\alpha)_{\alpha \in \Delta}$ which converges to $f \in B_{X_c^*}$ with respect to weak* topology on X^* . Thus, for $x \in X$ and $\epsilon > 0$ there exist some β_0 such that $\beta_0 \leq \beta$ and $|f(x) - f_{\alpha\beta}(x)| < \epsilon$. As a consequence of Theorem (2) we have that $|f_{\alpha\beta}(x)| \leq \|x\|_c$ for all $\beta \in \Lambda$, thus $|f(x)| < \epsilon + |f_{\alpha\beta}(x)| \leq \epsilon + \|x\|_c$ for all $\beta \geq \beta_0$. Hence $|f(x)| \leq \|x\|_c$ which implies that f is cone bounded, hence continuous. Thus $f \in B_{X_c^*}$ and that $B_{X_c^*}$ is compact with respect to weak* topology of X_c^* . But X is reflexive, which implies weak topology coincide with weak* topology since both are the weakest topology on $X = X^{**}$ making elements in X_c^* continuous. Thus, B_{X_c} is compact with respect to the weak topology of X_c .

" \Leftarrow " Assume that B_{X_c} is weakly compact. Let

$$J_c: X_c \rightarrow X_c^{**}$$

be the canonical embedding of X_c into X_c^{**} we know that J_c is linear and preserves the cone norm, thus it is continuous. Thus, also

$$J_c: X_{c(\tau_w)} \rightarrow X_{c(\tau_w^*)}^{**}$$

is continuous, where τ_w and τ_w^* are weak and weak topology respectively. By assumption $B_{X_c} \subseteq X_{c(\tau_w)}$ is compact, so the image $J_c(B_{X_c})$ is compact in $X_{c(\tau_w^*)}^{**}$. In particular $J_c(B_{X_c})$ is weak* closed. Thus, $J_c(B_{X_c}) = B_{X_c^{**}}$. As a consequence, $J_c(X_c) = X_c^{**}$. Hence X_c is reflexive.

4.0 CONCLUSION

The bi-dual of cone normed space was studied as an extension of the work of (Yusuf *et al.*, 2020) in which the dual of cone normed space was studied. Some properties in respect of the dual and bi-dual was established which allow us to define the reflexivity in the setting of cone normed space and proved the necessary and sufficient condition for cone normed space to be reflexive.

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