

Some Basic Properties of The Identity-Commuting Graph of Multigroups

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ABSTRACT

Multigroups are generalization of groups which are based on the multiset structure in which repetition of elements are allowed. The multisets were introduced to remedy the handicap idea of the classical set which does not allow repetitions of elements. In this paper, we study the multigroup G structure through its associated identity-commuting graph $\Gamma_{ic}(G)$ which is a simple graph whose vertices $V(\Gamma_{ic}(G))$ are the elements of the multigroup G and two distinct vertices x and y are adjacent if and only if $xy = yx = e$ and e is adjacent to all vertices of the graph, where e is the identity element. Some basic properties of this graph were studied which include: vertex degree, size, connectedness, completeness among others. It was shown that the graph is connected but not complete and condition for completeness was obtained.

Keywords: Multiset, Multigroup, Identity Graph, Commuting Graph, Complete Graph

1.0 INTRODUCTION

1.1 Introduce the Problem

There are several abstract algebra concepts which have been studied and facilitated by the construction of graphs. On the other hand, many properties of abstract algebraic structures have been characterized with the help of graph theory. Graphical representation of algebraic structures have been studied extensively and has become a research topic which is energizing over the recent couple of decades. These studies have raised several intriguing outcomes and a lot of research questions leading to more research areas and increasing interest by researchers (Ajay, Lavanya, Peter and Tamizh, 2021). The reason of allowing multiples of objects in a multiset (mset for short) is to address the real life situations where repetition is significant; such areas where multisets are applied include Biosystems (membrane and DNA computing), theoretical computer science, economics, language theory, social sciences etc (Isah and Tella, 2018).

A multiset M whose elements are drawn from a set X , is a set in which elements can occur more than once. The number of occurrence of an element x in M is called its multiplicity and denoted as $CM(x)$ (Costa, 2021). Let (X, \cdot) be a group, then the multigroup G over the group (X, \cdot) is a multiset whose elements are drawn from X and satisfies the following properties:

- $CG(x \cdot y) \geq \min\{CG(x), CG(y)\}$, $\forall x, y \in G$.
- $CG(x) = CG(x^{-1})$, $\forall x \in G$. (Gambo and Tella, 2022).

The identity graph of a multigroup G over a group (X, \cdot) is an undirected simple graph with G as vertex set and two distinct vertices x and y are connected if and only if $x \cdot y = y \cdot x = e$, where e is the identity of X . Similarly, the commuting graph of multigroup G over a group (X, \cdot) is an undirected simple graph with G as vertex set and two distinct vertices x and y are connected if and only if they commute. The identity graph of multigroups was discussed in (Sowaity, Sharada, and Naji, 2020) and (Sowaity, Sharada, and Naji, 2019) where basic properties like vertex degree, connectedness, clique, clique number, chromatic number, covering number etc were studied. Sowaity et al. (2019) obtained the condition that makes the identity graph of a multigroup to be complete, an Eulerian graph and a Hamiltonian graph. They also discussed the concept of twin multigroups and condition that makes two multigroups to be twin was obtained. Magami and Ashafa (2023) studied the commuting graph of multigroups where they showed that the multiplicity of the identity element is the maximum in a multigroup found the maximum degree of vertex in the graph and it is shown in (Magami and Ashafa, 2023) that, the commuting graph of a multigroup is a complete multipartite graph.

1.2 Motivation

Motivated by the work of Garba, Magami, and Ejima (2021), where they investigate some algebraic properties and combinatorial structures of the identity-power graph of group. In (Garba et al., 2021), the identity-power graph was created using two different graphs of group namely, the identity graph of groups

and the power graphs of groups. This paper is aimed at constructing a new graph of multigroup called the identity-commuting graph of multigroup using the algebraic properties of identity graph of multigroup discussed by Garba et al. (2021) and commuting graph of multigroup explored by Magami and Ashafa (2023) and study some of its basic properties.

1.3 Preliminaries

This section deals with some multigroup and graph theoretic terminologies together with some of the results obtained previously and applied to achieve our main results of this study. Moreover, the terminologies assist in the understanding of our paper. The graphs considered in this work are simple graphs; they are undirected graphs that do not have multiple edges or loops.

A graph Γ is formed by finite non-empty pair of sets V and E . V is the set of vertices which can be viewed as points while E is the set of edges which are line connecting these points. The number of elements in V is called the order of Γ and the total number of edges in Γ is called the size of Γ . $d(v)$ represents the degree of a vertex $v \in V$ in the graph Γ which is the number of edges connected to it (Garba et al., 2021). $\delta(\Gamma)$ represents the minimum vertex degree in a graph Γ while $\Delta(\Gamma)$ represents the maximum vertex degree in the graph Γ . Let $S \subseteq V$, then $\Gamma[S]$ is the induced subgraph of Γ and it is the graph with vertices set S and edges with both ends in S . A subset of V in which no two vertices are connected is called an independent set (Ajayi et al., 2021).

Definition 1. (Magami and Ashafa, 2023). Let X be a finite set, a multiset M over X is defined as a set that contains elements of X and allows an element to occur more than once in the multiset.

Example 1. Let $X = \{a, b, c, d, e\}$ then the following are examples of multisets over X :

- i. $M = [a, a, a, b, b, c, c, c, c, d, d, e, e, e, e, e]$
- ii. $M_2 = [a, b, c, c, d, d, e, e, e, e]$
- iii. $M_3 = [a, a, a, a, a, b, c, d, e, e]$
- iv. $M_4 = [a, b, c, d, e, e, e, e, e]$

The number of repetition of an element in a multiset is known as its multiplicity and we write $C_M(a) = 3$ to mean that the multiplicity of the element a in the multiset (mset for short) M is 3. Observe that, elements of the multiset are written inside square brackets, this is done to differentiate it from the classical set (Gambo

and Tella, 2022). However, there are many other different ways for representing multiset for instance the multiset $M_4 = [a, b, c, d, e, e, e, e, e]$ can be represented as $M_4 = [a_1, b_1, c_1, d_1, e_5]$ with subscripts of an elements representing its multiplicity. $MS(X)$ denotes the set of all multisets over X .

Definition 2. Let $M \in MS(X)$, a subset S of X is called the support or root of the mset M if for every $x \in M$ with $C_M(x) > 0$, $\exists x \in S$ and for every $x \in M$ with $C_M(x) = 0$, $\exists x \notin S$ (Gambo and Tella, 2022).

Example 2. Let $M = [a_4, b_2, c_8, d_1]$, then clearly $S = \{a, b, c, d\}$ is a support of M .

Definition 3. A multiset M over X is said to be r -regular if all its objects have the same multiplicity and the common multiplicity is r ; M is irregular if otherwise (Gambo and Tella, 2022).

Example 3. Let $M = [a_4, b_4, c_4, d_4]$, then M is a 4-regular multiset.

Definition 4. Let $(X, *)$ be a group and $G \in MS(X)$, then G is a known as a multigroup (mgroup) over X if $\forall x, y \in X$ we have:

1. $C_G(x * y) \geq C_G(x) \cap C_G(y)$ where $C_G(x) \cap C_G(y) = \min\{C_G(x), C_G(y)\}$
2. $C_G(x^{-1}) = C_G(x)$

The set of all mgroup over the group X is denoted by $MG(X)$ (Ejegwa and Ibrahim, 2020).

Example 4. Let $(X, *) = (Z_4, \oplus_4)$ then $G = [0_3, 1_2, 2_3, 3_2]$ is a mgroup over Z_4 .

Definition 5. Let $(X, *)$ be group, then the Commuting Graph of the multigroup G , $\Gamma_{com}(G)$, is the graph whose vertices are the elements of G and two distinct vertices $x, y \in X$ are connected if they commute, that is if $x * y = y * x$ (Magami and Ashafa, 2023).

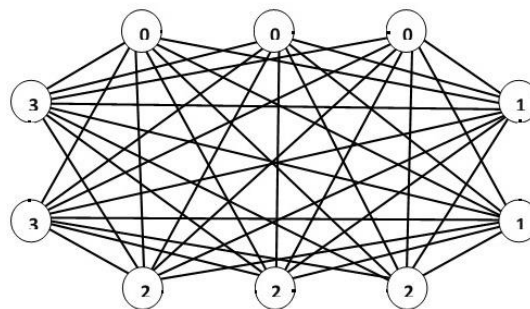


Figure 1: Commuting Graph of multigroup G

Example 5. The Commuting Graph of multigroup $G = [0, 0, 0, 1, 1, 2, 2, 2, 3, 3]$ over Z_4 is figure 1.

Definition 6. Let $(X,*)$ be group, then the Identity Graph of the multigroup G , $\Gamma_{id}(G)$, is the graph whose vertices are the elements of G and two distinct vertices $x, y \in X$ are connected if $x * y = e$ and e is connected to all other vertices, e is the identity element of X (Sowaity et al., 2020).

Example 6. The Identity Graph of multigroup $G = [0, 0, 0, 1, 1, 2, 2, 2, 3, 3]$ over Z_4 is figure 2.

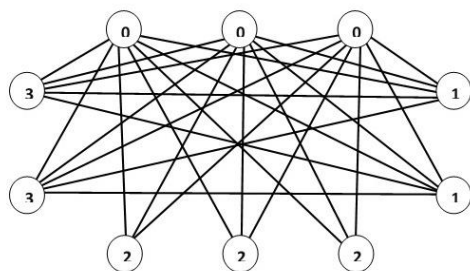


Figure 2: Identity Graph of multigroup G

Definition 7. A graph Γ is said to be complete, if every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n (Ajay et al., 2021).

Definition 8. A graph Γ is said to be connected if any two distinct vertices of Γ are joined by a path. Γ is said to be disconnected if Γ is not connected (Ajay et al., 2021).

Definition 9. (Self Inverse Elements). Let $(X,*)$ be a finite Abelian group, an element $x \in X$ is called self inverse element if $x^{-1} = x$. The set of all self inverse elements in X is denoted by S_{inv} (Sowaity et al., 2020).

Definition 10. (Mutual Inverse Elements). Let $(X,*)$ be a finite Abelian group, an element $y \in X$ is called mutual inverse element if $\exists y' \in X$ with $y \neq y'$ such that $y^{-1} = y'$. The set of all mutual inverse elements in X is denoted by M_{inv} (Sowaity et al., 2020).

2.0 REPRESENTATION OF MULTIGROUPS

Notation 2.1. Let $(X,*)$ be a finite Abelian group where $X = \{x_1, x_2, x_3, \dots, x_n\}$ with $x_1 = e$ is the group identity element. $|X|$ represent the cardinality of X .

Notation 2.2. Let $(X, *)$ be a finite Abelian group where $|X| = n$ with $x_1 = e$, then elements of X can be partitioned into two subsets namely self inverse (S_{inv}) and mutual inverse (M_{inv}) elements (Magami and Ibrahim, 2021). Let $|S_{inv}| = k$ then $|M_{inv}| = n - k$. Since $X = \{x_1, x_2, x_3, \dots, x_n\}$ then, we let $S_{inv} = \{x_1, x_2, x_3, \dots, x_k\}$ and $M_{inv} = \{x_{k+1}, x_{k+2}, x_{k+3}, \dots, x_n\}$

$\dots, x_n\}$

Also, M_{inv} can further be partitioned into two equal subsets, the first containing the mutual inverse elements and the other containing their inverses (Usman and Magami, 2015), (Magami, Amama, and Garba, 2022). So, we have;

$$M_{inv} = \{x_{k+1}, x_{k+2}, x_{k+3}, \dots, x_{(n-k)/2}, (x_{k+1})^{-1}, (x_{k+2})^{-1}, (x_{k+3})^{-1}, \dots, (x_{(n-k)/2})^{-1}\}$$

Therefore, let

$$M_{\Delta} = \{x_{k+1}, x_{k+2}, x_{k+3}, \dots, x_{(n-k)/2}\} \text{ and } M_{\Delta}^{-1} = \{(x_{k+1})^{-1}, (x_{k+2})^{-1}, (x_{k+3})^{-1}, \dots, (x_{(n-k)/2})^{-1}\}$$

$$\text{then } |M_{\Delta}| = |M_{\Delta}^{-1}| = (n - k)/2 \text{ with } |M_{inv}| = |M_{\Delta}| + |M_{\Delta}^{-1}| = n - k$$

Notation 2.3. Let G be a multigroup over the finite Abelian group X , then we have the following notations:

1. $x \sim y$ denotes that the 2 vertices x and y are adjacent, and given a set of vertices V , then we write $x \sim V$ if a the vertex v is adjacent to every vertex in a set V .
2. $MG(X)$ denotes the set of all multigroups over the group X
3. Square brackets ($[]$) will be used to list elements of msets.
4. $G = [X_1, X_2, X_3, \dots, X_n]$ represents the multigroup G over the group X where $X_i = [x_i, x_i, x_i, \dots, x_i]$ is the mset containing repeated $x_i \in X$ and $|X_i|$ denotes the cardinality of X_i which is the also the count function or multiplicity of x_i in the multigroup G i.e. $|X_i| = C_G(x_i)$. Note that X_1 is the mset which contains repeated identity element $x_1 = e$ of the group X .
5. Given a multigroup $G = [X_1, X_2, X_3, \dots, X_n]$, then the msets $[X_1, X_2, X_3, \dots, X_k]$ are msets each contains repeated self inverse elements while the msets $[X_{k+1}, X_{k+2}, X_{k+3}, \dots, X_n]$ are mset each contains mutual inverse elements. Let

$$S_{inv}(G) = [X_1, X_2, X_3, \dots, X_k] \text{ and}$$

$$M_{inv}(G) = [X_{k+1}, X_{k+2}, X_{k+3}, \dots, X_n]$$

6. $M_{inv}(G) = [X_{k+1}, X_{k+2}, X_{k+3}, \dots, X_n]$ for the multigroup G can be partitioned in the same manner we partitioned M_{inv} for the group X . Thus we let $M_{\Delta}(G) = [X_{k+1}, X_{k+2}, X_{k+3}, \dots, X_{(n-k)/2}]$ and $M_{\Delta}^{-1}(G) = [(X_{k+1})^{-1}, (X_{k+2})^{-1}, (X_{k+3})^{-1}, \dots, (X_{(n-k)/2})^{-1}]$ with $|M_{inv}(G)| = |M_{\Delta}(G)| + |M_{\Delta}^{-1}(G)|$ and elements in $(X_j)^{-1}$ are inverses of elements in X_j where $j, j^{-1} \in \{k + 1, k + 2, k + 3, \dots, n\}$.

7. The cardinality of a multigroup G is given by $|G| = \sum_{i=1}^n |X_i|$

3.0 MAIN RESULTS ON IDENTITY-



COMMUTING GRAPH OF MULTIGROUPS

3.1 Identity-Commuting Graph of Multigroups: Let X be a group and G be a multigroup over X , we define the Identity-Commuting graph of the multigroup G , $\Gamma_{ic}(G)$, as the simple graph whose vertices are precisely the elements of G and two distinct vertices $x, y \in G$ are connected ($x \sim y$) if and only if either they commute, that is if $x * y = y * x$ or $x * y = e$ and e is connected to all other vertices, e is the identity element of X . From the above definition of Identity-Commuting graph of multigroup, we can compact the definition as follows: Let X be a group and G be a multigroup over X , we define the Identity-Commuting graph of the multigroup G , $\Gamma_{ic}(G)$, as the simple graph whose vertices are precisely the elements of G and two distinct vertices $x, y \in G$ are connected ($x \sim y$) if and only if $x * y = y * x = e$ and e is connected to all other vertices, e is the identity element of X .

Theorem 3.1. Let X be a finite Abelian group and G be a multigroup over X , then

- i. $\Gamma_{ic}(G)$ is always connected.
- ii. $\Gamma_{ic}(G)$ never complete if $\exists x_i, x_j \in X$ with $x_i \in S_{inv} \setminus \{e\}$, $x_j \in M_{inv}$
- iii. Let $x, y \in G$ with $x \neq y$, then $x \sim y$ iff either $x = e$ or $y = e$ or $x * y = e$ where e is the identity element.

Proof. Let X be a finite Abelian group and G be a multigroup over X , then

1. Let $x, y \in V(\Gamma_{ic}(G))$, Then we have the following cases:

- Case 1** ($x, y \in X_1$): Since $x * y = e * e = e$, then $x \sim y$.
- Case 2** (Either $x \in X_1, y \notin X_1$ or $y \in X_1, x \notin X_1$): Since $e \sim v \forall v \in G$, then $x \sim y$.
- Case 3** ($x, y \notin X_1$): Since $e \sim v \forall v \in G$, then $x \sim e \sim y$.

Since x and y are arbitrary, then $\Gamma_{ic}(G)$ is connected.

2. It suffice to show that there exists 2 vertices that are not connected. Let $x, y \in V(\Gamma_{ic}(G))$ such that $x \in X_i, i \in \{2, 3, 4, \dots, k\}$ and $y \in X_j, j \in \{k + 1, k + 2, k + 3, \dots, n\}$.

Claim: x is not adjacent to y

Proof. Since $y \neq x^{-1}$ because $x = x^{-1}$ and also $y \neq e$ because $y \in M_{inv}(G)$, then x and y are not connected. Therefore, $\Gamma_{ic}(G)$ contains disconnected vertices and hence never complete.

3. Let $x, y \in G$ with $x \neq y$ and assume $x \sim y$ then by definition of $\Gamma_{ic}(G)$ either $x = e$ or $y = e$ since e is connected to every vertex of $\Gamma_{ic}(G)$, or $x * y = e$.

Conversely, let $x, y \in G$ with $x \neq y$ and assume $x = e$ or $y = e$ or $x * y = e$, then we have the following cases:

- Case 1** ($x = e$ or $y = e$): then $x \sim y$, since e is connected to every vertex of $\Gamma_{ic}(G)$.
- Case 2** ($x * y = e$): then $x \sim y$ by definition of $\Gamma_{ic}(G)$.

Theorem 3.2. Let X be a finite Abelian group and G be a multigroup over X . Let $v \in V(\Gamma_{ic}(G))$ then

1. The degree of v is given by

$$d(v) = \begin{cases} |G| - 1, & v \in X_1 \\ |X_1| + |X_i| - 1, & v \in X_i, i \in \{2, 3, 4, \dots, k\} \\ |X_1| + |X_j|, & v \in X_j, j \in \{k + 1, k + 2, k + 3, \dots, n\} \end{cases}$$
2. $\delta(G) = \min\{d(v)\} = |X_1| + \min \left\{ \begin{matrix} |X_i| - 1, & i \in \{2, 3, 4, \dots, k\} \\ |X_j|, & j \in \{k + 1, k + 2, k + 3, \dots, n\} \end{matrix} \right\}$

3. $\Delta(G) = \max\{d(v)\} = |G| - 1$

Proof. Let X be a finite Abelian group and G be a multigroup over X . Let $v \in V(\Gamma_{ic}(G))$ then

1. If $v \in V(\Gamma_{ic}(G)) = G = [X_1, X_2, X_3, \dots, X_n]$, then we have the following 3 cases:

- Case 1** ($v \in X_1$): Then v is connected to every other vertex except itself. That is

$$v \sim x, \forall x \in V(\Gamma_{ic}(G)) \setminus \{v\}$$

$$\Rightarrow d(v) = |G| - 1$$

Case 2 ($v \in X_i, i \in \{2, 3, 4, \dots, k\}$): $[X_i]$ is the mset containing repeated v . Let $[(X_i)^{-1}]$ be the mset containing repeated v^{-1} , then $|X_i| = |[(X_i)^{-1}]|$ because $v = v^{-1}$. Since $v \neq e$ and $v = v^{-1}$, then by definition of $\Gamma_{ic}(G)$, v is connected to all vertices in $[X_i]$ and also connected to all vertices in $[(X_i)^{-1}] = [X_i]$ (since $v = v^{-1}$) except itself. Thus,

$$v \sim [X_i] \setminus \{v\} \text{ and } \Rightarrow d(v) = |X_1| + |X_i| - 1$$

Case 3 ($v \in X_j, j \in \{k + 1, k + 2, k + 3, \dots, n\}$): $[X_j]$ is the mset containing repeated v . Let $[(X_j)^{-1}]$ be the mset containing repeated v^{-1} , then $|X_j| = |[(X_j)^{-1}]|$ by definition of multigroup and $v \neq v^{-1}$. Since $v \neq e$ and $v \neq v^{-1}$, then by definition of $\Gamma_{ic}(G)$, v is connected to all vertices in X_1 and also connected to all vertices in $(X_j)^{-1}$. Thus,

$$v \sim [X_1] \text{ and } v \sim [(X_j)^{-1}] \Rightarrow d(v) = |X_1| + |[(X_j)^{-1}]| = |X_1| + |X_j| \text{ since } |X_j| = |[(X_j)^{-1}]|$$

2. $\delta(G) = \min\{d(v)\}$. From (1) above we have



$$\begin{cases} |G| - 1, & v \in X_1 \\ |X_1| + |X_i| - 1, & v \in X_i, i \in \{2, 3, 4, \dots, k\} \\ |X_1| + |X_j|, & v \in X_j, j \in \{k + 1, k + 2, k + 3, \dots, n\} \end{cases}$$

Therefore,

$$\delta(G) = \min\{d(v)\} = \min\{|G| - 1, |X_1| + |X_i| - 1, |X_1| + |X_j|\}$$

$$= |X_1| + \min\{|X_i| - 1, |X_j|\}$$

Since $|G| > |X_1| + |X_i|$ and $|G| > |X_1| + |X_j|$

3. $\Delta(G) = \max\{d(v)\}$. From (1) above we have

$$\Delta(G) = \max\{d(v)\} = \max\{|G| - 1, |X_1| + |X_i| - 1, |X_1| + |X_j|\}$$

$$= |G| - 1$$

Since $|G| > |X_1| + |X_i|$ and $|G| > |X_1| + |X_j|$

Theorem 3.3. Let X be a finite Abelian group and G be a multigroup over X . Let $H \subseteq G$ then the induced subgraph $\Gamma[H]$ of $\Gamma_{ic}(G)$ is complete iff H satisfies any of the following conditions:

- i. $H = X_i, i \in \{1, 2, 3, 4, \dots, k\}$
- ii. $H = X_1 \cup X_i, i \in \{2, 3, 4, \dots, k\}$
- iii. $H = X_j \cup (X_j)^{-1}, j \in \{k + 1, k + 2, k + 3, \dots, n\}, |X_j| = 1$.
- iv. $H = X_1 \cup X_j \cup (X_j)^{-1}, j \in \{k + 1, k + 2, k + 3, \dots, n\}, |X_j| = 1$.

Proof. (\Rightarrow) Let X be a finite Abelian group and G be a multigroup over X . Let $H \subseteq G$ and assume that the induced subgraph $\Gamma[H]$ of $\Gamma_{ic}(G)$ is complete. We will analyze the subsets of G for which every element in the subset is connected to all other elements of the subset. Consider the multigroup $G = [X_1, X_2, X_3, \dots, X_n]$ which can be partitioned into self inverse and mutual inverse msets, that is $G = [X_1, X_2, X_3, \dots, X_k, X_{k+1}, X_{k+2}, \dots, X_n]$, the self inverse msets $S_{inv}(G) = [X_1, X_2, X_3, \dots, X_k]$ and mutual inverse msets $M_{inv}(G) = [X_{k+1}, X_{k+2}, \dots, X_n]$.

Firstly, let $X_i \subset S_{inv}(G)$. Pick $x, y \in X_i$ with $x \neq y$, then $x * y = e$ because elements in the mset X_i are repeated self inverse elements and by definition of $\Gamma_{ic}(G)$, $x \sim y$ and since x and y are arbitrary, this means that $\Gamma[X_i], i \in \{1, 2, 3, \dots, k\}$ is complete. Therefore, $H = X_i, i \in \{1, 2, 3, \dots, k\}$ which proves part (1).

Secondly, all elements of the mset X_1 are connected to all other elements in G by definition of $\Gamma_{ic}(G)$. Since all vertices in X_1 are connected to all vertices in $X_i, i \in \{2, 3, \dots, k\}$ and also, vertices in both X_1 and each $X_i, i \in \{2, 3, \dots, k\}$ have been proven above to be

connected to each other, then the subgraph $\Gamma[X_1 \cup X_i], i \in \{2, 3, \dots, k\}$ is complete. Therefore, $H = X_1 \cup X_i, i \in \{2, 3, \dots, k\}$ which proves part (2).

Thirdly, analyzing the mutual inverse elements, $X_j, j \in \{k + 1, k + 2, k + 3, \dots, n\}$, we find out that if $|X_j| > 1$ then elements in X_j are disconnected because suppose $x, y \in X_j$ then $x * y \neq e$ and hence x not adjacent to y . However, all elements in X_j are connected to all elements in $(X_j)^{-1}, j^{-1} \in \{k + 1, k + 2, k + 3, \dots, n\}$. Then the subgraph $\Gamma[X_j \cup X_j^{-1}]$ will only be complete if $|X_j| = |X_j^{-1}| = 1$. Therefore, $H = X_j \cup X_j^{-1}, j \in \{k + 1, k + 2, k + 3, \dots, n\}, |X_j| = |X_j^{-1}| = 1$, which proves part (3).

Finally, the subgraph $\Gamma[X_1 \cup X_j \cup X_j^{-1}], j \in \{k + 1, k + 2, k + 3, \dots, n\}, |X_j| = |X_j^{-1}| = 1$ is complete since by definition of $\Gamma_{ic}(G)$, all elements X_1 are connected to all elements of G . Therefore, $H = X_1 \cup X_j \cup X_j^{-1}, j \in \{k + 1, k + 2, k + 3, \dots, n\}, |X_j| = |X_j^{-1}| = 1$ which proves part (4).

(\Leftarrow) Conversely, let X be a finite Abelian group and G be a multigroup over X . Let $H \subseteq G$ and assume H satisfies any of the following conditions:

- i. $H = X_i, i \in \{1, 2, 3, 4, \dots, k\}$
- ii. $H = X_1 \cup X_i, i \in \{2, 3, 4, \dots, k\}$
- iii. $X_j \cup X_j^{-1}, j \in \{k + 1, k + 2, k + 3, \dots, n\}, |X_j| = 1$.
- iv. $H = X_1 \cup X_j \cup X_j^{-1}, j \in \{k + 1, k + 2, k + 3, \dots, n\}, |X_j| = 1$.

Then, for each H , we will prove that $\Gamma[H]$ is complete.

1. Let $x, y \in H = X_i, i \in \{1, 2, 3, 4, \dots, k\}$ with $x \neq y$, then $x * y = e$. This means $x \sim y$ and since x and y are arbitrary, then $\Gamma[H]$ is complete.

2. Let $x, y \in H = X_1 \cup X_i, i \in \{2, 3, 4, \dots, k\}$ with $x \neq y$, then we have the following cases,

Case 1 ($x, y \in X_1$): $\Gamma[H]$ is complete by 1 above.

Case 2 ($x, y \in X_i, i \in \{2, 3, 4, \dots, k\}$): $\Gamma[H]$ is complete by 1 above.

Case 3 ($x \in X_1, y \in X_i, i \in \{2, 3, 4, \dots, k\}$ or $y \in X_1, x \in X_i, i \in \{2, 3, 4, \dots, k\}$): $\Gamma[H]$ is complete because $x \sim y$ by definition of $\Gamma_{ic}(G)$.

3. Let $x, y \in X_j \cup X_j^{-1}, j \in \{k + 1, k + 2, k + 3, \dots, n\}, |X_j| = 1$. Since $|X_j| = |X_j^{-1}| = 1$, then either $x \in X_j, y \in X_j^{-1}$ or $y \in X_j, x \in X_j^{-1}$ which in any case $x * y = e$, thus $x \sim y$. Since x and y are arbitrary, then $\Gamma[H]$ is complete.



4. Let $x, y \in H = X_1 \cup X_j \cup X_j^{-1}, j \in \{k + 1, k + 2, k + 3, \dots, n\}, |X_j| = 1$, then we have the following cases,

Case 1 ($x, y \in X_1$): $\Gamma[H]$ is complete by 1 above.

Case 2 ($x \in X_j, y \in X_{j-1}$ or $y \in X_j, x \in X_{j-1}$): $\Gamma[H]$ is complete by 3 above.

Case 3 ($x \in X_1, y \in X_j$ or $x \in X_1, y \in X_{j-1}$): $\Gamma[H]$ is complete because $x \sim y$ by definition of $\Gamma_{ic}(G)$ every element of X_1 is connected to all elements in G .

Case 4 ($y \in X_1, x \in X_j$ or $y \in X_1, x \in X_{j-1}$): $\Gamma[H]$ is complete because $x \sim y$ by definition of $\Gamma_{ic}(G)$ every element of X_1 is connected to all elements in G .

Lemma 3.1. Let $G = \{e, a, b\}$ be a set with 3 elements, then G is an Abelian group under the operation defined by the following table

*	e	a	b
e	e	a	b
a	a	a*a	e
b	b	b	b*b

and $a * a = b$

Theorem 3.4. Let X be a finite Abelian group and G be a multigroup over X . Then $\Gamma_{ic}(G)$ is complete iff any of the following conditions are satisfied:

- i. $X = \{x_1\}$ where $x_1 = e$
- ii. $X = \{x_1, x_2\}, x_1 = e, x_2 = (x_2)^{-1}$
- iii. $X = \{x_1, x_2, x_3\}, x_1 = e, (x_2)^{-1} = x_3$ with $|X_2| = |X_3| = 1$ and $(x_2)^2 = x_3$

Proof. Let X be a finite Abelian group and G be a multigroup over X .

(\Rightarrow) Assume $\Gamma_{ic}(G)$ is complete, then by theorem 3.5, the subgraph $\Gamma[H]$ of the identity-commuting graph of a multigroup is complete iff its vertices H satisfy any of the following:

- i. 1. $H = X_i, i \in \{1, 2, 3, 4, \dots, k\}$
- ii. 2. $H = X_1 \cup X_i, i \in \{2, 3, 4, \dots, k\}$
- iii. 3. $H = X_j \cup X_{j-1}, j \in \{k + 1, k + 2, k + 3, \dots, n\}, |X_j| = 1$.
- iv. 4. $H = X_1 \cup X_j \cup X_j^{-1}, j \in \{k + 1, k + 2, k + 3, \dots, n\}, |X_j| = 1$.

Since $\Gamma[H]$ is complete if H satisfies any of the above condition and H being a subset of a multigroup does not make it a multigroup, then we need to check which among these condition will guarantee H to be a

multigroup which will then make $\Gamma[H]$ a complete identity-commuting graph of a multigroup. So we will be checking whether the support of H (the set where the elements of H are gotten from) is a group; if this support is a group then H becomes a multigroup otherwise H is not a multigroup but just a subset of a multigroup.

Lets start with the first condition

$$H = X_i, i \in \{1, 2, 3, 4, \dots, k\}$$

The support of H has only one element which is $\{x_i\}$ with $x_i \in X, i \in \{1, 2, 3, 4, \dots, k\}$. $\{x_i\}$ can only be a group if $i = 1$. Therefore $\Gamma_{ic}(X_1)$ is complete. This proves part (1).

For the second condition,

$$H = X_1 \cup X_i, i \in \{2, 3, 4, \dots, k\}$$

□

The support of H has only 2 elements $\{x_1, x_i\}$ with $x_i \in X, i \in \{2, 3, 4, \dots, k\}$. Since $x_i, i \in \{2, 3, 4, \dots, k\}$ in the support is self inverse, it is clear that $\{x_1, x_i\}$ is a group which makes H a multigroup. Hence we have the support as $X = \{x_1, x_2\}$ with $x_1 = e$ and $x_2 = (x_2)^{-1}$. Therefore $\Gamma_{ic}(X_1 \cup X_2)$ is complete. This proves part (2).

In the third condition

$$H = X_j \cup X_j^{-1}, j \in \{k + 1, k + 2, k + 3, \dots, n\}, |X_j| = 1$$

The support of H has only 2 elements $\{x_j, (x_j)^{-1}\}, j \in \{k + 1, k + 2, k + 3, \dots, n\}$ which are mutual inverse elements and hence cannot be a group because it does not contain the identity element. Therefore, H cannot be a multigroup.

Finally, the fourth condition

$$H = X_1 \cup X_j \cup X_j^{-1}, j \in \{k + 1, k + 2, k + 3, \dots, n\}, |X_j| = 1$$

The support of H has only 3 elements which is $\{x_1, x_j, (x_j)^{-1}\}$ with $j, j^{-1} \in \{k + 1, k + 2, k + 3, \dots, n\}, |X_j| = 1$. The support can be expressed as $X = \{x_1, x_2, x_3\}$ with $(x_2)^{-1} = x_3$ and by Lemma 3.1 is an Abelian group if $(x_2)^2 = x_3$. Therefore, $\Gamma_{ic}(X_1 \cup X_j \cup X_j^{-1})$ with $j, j^{-1} \in \{k + 1, k + 2, k + 3, \dots, n\}, |X_j| = 1$ is complete. This proves part (3).

Q Assume that

- i. $X = \{x_1\}$ where $x_1 = e$
- ii. $X = \{x_1, x_2\}, x_1 = e, x_2 = (x_2)^{-1}$
- iii. $X = \{x_1, x_2, x_3\}, x_1 = e, (x_2)^{-1} = x_3$ with $|X_2| = |X_3| = 1$ and $(x_2)^2 = x_3$

then, our goal is to prove that $\Gamma_{ic}(G)$ is complete where $G \in MG(X)$.

Condition 1: Let $X = \{x_1\}$ where $x_1 = e$, then $G = [X_1]$. Let $x, y \in V(\Gamma_{ic}(G)) = G$ with $x \neq y$. Since $x, y \in X_1$

then by definition of $\Gamma_{ic}(G)$, $x \sim y$. Hence $\Gamma_{ic}(G)$ is complete since x and y are arbitrary.

Condition 2: Let $X = \{x_1, x_2\}$, $x_1 = e$, $x_2 = (x_2)^{-1}$, then $G = [X_1, X_2]$. Let

$x, y \in V(\Gamma_{ic}(G)) = G$ with $x \neq y$. We have the following cases

1. If $x, y \in X_1$ then $x \sim y$ by condition 1 above. Hence $\Gamma_{ic}(G)$ is complete since x and y are arbitrary.
2. If $x \in X_1, y \in X_2$ or $y \in X_1, x \in X_2$ then $x \sim y$ because by definition of $\Gamma_{ic}(G)$ every vertex in X_1 is connected to every other vertex in G . Hence $\Gamma_{ic}(G)$ is complete since x and y are arbitrary.
3. $x, y \in X_2$ then $x \sim y$ because $x * y = e$. Hence $\Gamma_{ic}(G)$ is complete since x and y are arbitrary.

Condition 3: Let $X = \{x_1, x_2, x_3\}$, $x_1 = e$, $(x_2)^{-1} = x_3$ with $|X_2| = |X_3| = 1$ and $(x_2)^2 = x_3$, then $G = [X_1, x_2, x_3]$. Let $x, y \in V(\Gamma_{ic}(G)) = G$ with $x \neq y$. We have the following cases

- i. If $x, y \in X_1$ then $x \sim y$ by condition 1 above. Hence $\Gamma_{ic}(G)$ is complete since x and y are arbitrary.
- ii. If $x \in X_1, y \in \{x_2, x_3\}$ or $y \in X_1, x \in \{x_2, x_3\}$ then $x \sim y$ because by definition of $\Gamma_{ic}(G)$ every vertex in X_1 is connected to every other vertex in G . Hence $\Gamma_{ic}(G)$ is complete since x and y are arbitrary.
- iii. If $x = x_2, y = x_3$ or $y = x_2, x = x_3$ then $x \sim y$ because $x * y = e$. Hence $\Gamma_{ic}(G)$ is complete since x and y are arbitrary.

Theorem 3.5. Let X be a finite Abelian group and G be a multigroup over X . Then the total number of edges of $\Gamma_{ic}(G)$ is given by

$$|E(\Gamma_{ic}(G))| = \sum_{i=1}^k \frac{|X_i|(|X_i| - 1)}{2} + \frac{1}{2} \cdot \sum_{j=k+1}^n |X_j|^2 + |X_1| \cdot \sum_{i=2}^n |X_i|$$

Proof. Let X be a finite Abelian group and G be a multigroup over X . Then the total number of edges of $\Gamma_{ic}(G)$ can be partitioned into the following sets:

1. Set of edges of subgraphs induced by the mset of self inverse elements of X
 $S_{inv}(G) = [X_1, X_2, X_3, \dots, X_k]$
 By definition of $\Gamma_{ic}(G)$, if $x, y \in X_i, i \in \{1, 2, 3, \dots, k\}$ then $x \sim y$
2. Set of edges connecting mutual inverse elements with their inverses
 $M_{inv}(G) = [X_{k+1}, X_{k+2}, X_{k+3}, \dots, X_n]$
 which contains 2 sets

$M_{\Delta}(G) = [X_{k+1}, X_{k+2}, X_{k+3}, \dots, X_{(n-k)/2}]$
 $M_{\Delta}^{-1}(G) = [(X_{k+1})^{-1}, (X_{k+2})^{-1}, (X_{k+3})^{-1}, \dots, (X_{(n-k)/2})^{-1}]$
 For any $x_j \in X_j, j \in \{k+1, k+2, k+3, \dots, n\}$ there exists $(x_j)^{-1} \in X_j^{-1} \neq X_j, j^{-1} \in \{k+1, k+2, k+3, \dots, n\}$. Since $x_j * (x_j)^{-1} = e$ then $x_j \sim (x_j)^{-1}$.

3. Set of edges connecting vertices in X_1 to all vertices in every other mset in G .
 The sum of the above 3 sets of edges gives the total number of edges in $\Gamma_{ic}(G)$. We now find the edges in each set and later sum them up.

1. Since $\Gamma[X_i], i \in \{1, 2, 3, \dots, k\}$ is complete then the first set of edges is

$$|E(\Gamma[X_i])| = \sum_{i=1}^k \frac{|X_i|(|X_i| - 1)}{2} \quad \dots (1)$$

2. For every $X_j \in M_{inv}(G)$ there exists $X_j^{-1} \in M_{inv}(G)$ such that $\Gamma[X_j \cup X_j^{-1}]$ is a complete bipartite graph with 2 independent set of vertices X_j and X_j^{-1} . Therefore the total edges in $\Gamma[X_j \cup X_j^{-1}]$ is given by

$$|E(\Gamma[X_j \cup X_j^{-1}])| = E(K_{|X_j|, |X_j^{-1}|}) = E(K_{|X_j|, |X_j|}) = \frac{1}{2} \cdot \sum_{j=k+1}^n |X_j|^2 \quad \dots (2)$$

since $|X_j| = |X_j^{-1}|$

3. Edges connecting vertices in X_1 to all vertices in every other mset in G is given by

$$|X_1| \cdot \sum_{i=2}^n |X_i| \quad \dots (3)$$

Summing up equations (1), (2) and (3), yields the required result.

4 CONCLUSION

In this investigation, we investigated the graph theoretic properties of identity-commuting graph of multigroups. The identity-commuting graph of multigroups is a graph created by merging the properties of two different graphs namely, the identity graph of multigroups and the commuting graph of multigroups. The findings and conclusion taken from the results obtained are highlighted below.

The vertex degree of the identity-commuting graph of multigroup vertex was obtained by classifying the vertex set into 3 partitions. Firstly, if the vertex is the identity element then the degree is 1 less than the cardinality of the vertex set. Secondly, if the vertex is a self inverse element which is not the identity element then its degree is the cardinality of the mset containing repeated identity elements plus the mset containing the vertex minus 1 and thirdly, if the vertex is a mutual inverse element then its degree is the cardinality of the

mset containing repeated identity elements plus the mset containing the vertex. The maximum degree was found to be equal to the degree of the identity element while the minimum degree was found to be the cardinality of the mset containing repeated identity element plus the minimum among these two sets, 1 less than the cardinality of mset containing a self inverse element which is not the identity and the cardinality of the mset containing a mutual inverse element.

It was shown that the identity-commuting graph is always connected. Therefore the graph do not contain an isolated vertex. This also means that the graph can never be null graphs since it must contain edges which connects the identity element to all the vertices.

The completeness of the graph was investigated and found not to be "automatic" (not without some conditions). For the identity-commuting graph, the condition for its completeness depends on the elements set of its support group which are; either the support group has only one element or it has only two elements which are all self inverse or it has only 3 elements with two among these elements mutual inverses each of whose cardinality of its msets in the multigroup is equal to 1. The expressions of the sizes of the graph was found.

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