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Maclaurin Series Solution for Seventh Order Boundary Value Problems

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ABSTRACT

Maclaurin series expansion which is taught at undergraduate levels and which has been previously used for the solution of initial value problem is successfully used in this paper for the solutions of seventh-order boundary value problems. The governing equation defined between boundary points 0 and 1 is differentiated to get high order derivatives and the assumed solution is expanded in Maclaurin series then, the boundary conditions at $x=1$ are utilized to determine unknown coefficients. The method is simple and the numerical results displayed on the tables were found to be in good agreement with the solution and even better than some existing results in the literature. This agreement is also confirmed from the figures where the graph of exact solution and approximate solution overlap.

Keywords: Boundary value problem, ordinary differential equation, seventh order, system of algebraic equations, Maclaurin series.

1.0 INTRODUCTION

Boundary Value Problems (BVPs) are encountered in various fields of science and engineering. They involve finding solutions to differential equations while satisfying certain conditions at the boundaries of the domain. These problems play a crucial role in understanding and modeling various physical phenomena, ranging from fluid flow and heat transfer to electromagnetic fields and quantum mechanics (Raji *et al.*, 2023). It's true that solving BVPs analytically can be challenging, leading to the use of numerical methods. Many authors have contributed to this field such as: Siddiqi & Iftikhar (2015) developed variational iteration method using He's polynomials for numerical solution of seventh order boundary value problems. Viswanadham & Reddy (2015) applied a finite element method involving Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions for the solution of seventh order BVPs. While Reddy (2016) proposed a finite element method involving collocation method with sextic B-splines as basis functions for numerical solution of linear and nonlinear seventh order BVPs.

In Al- Mamun & Asaduzzaman 2019 the authors introduced some basic idea of Variational iteration

method for the numerical solution of seventh order BVPs. Pandey, (2022) applied a novel finite difference method, derived from the splitting coupled equations approach for the numerical solution of seventh order BVPs. Olumide *et al.*, (2020) proposed 7th order continuous block methods for approximate solution of sixth-order BVPs. Further, Arifeen *et al.*, (2021) developed Haar wavelet method for the approximate solutions of higher order linear and nonlinear BVPs. Tsetimi *et al.*, (2022) considered modified variational iteration method by using Chebyshev polynomials to solve 12th order BVPs. He and Ji (2019) applied Taylor series to solve Lane Emden equation. He (2020) developed the idea of using Taylor series to solve third - order BVPs appearing in Architectural engineering. Motivated by He, (2020) we studied numerical solution of seventh order BVPs by using Maclaurin expansion.

The paper is organized as follows. In Section 2, the methodology of the proposed method is outlined, and numerical examples are given in Section 3 for linear and nonlinear BVPs to present the accuracy and simplicity of the method, while conclusion is addressed in the Section 4.

2.0 Methodology

In this paper, linear and nonlinear seventh-order BVPs are considered in the following form:

$$y^{(7)}(x) = f(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)}, y^{(6)}) \quad 0 < x < 1 \quad (2.1)$$

Subject to the following boundary conditions:

$$\left. \begin{aligned} y(0) &= \alpha_0 & y(1) &= \beta_0 \\ y^{(1)}(0) &= \alpha_1 & y^{(1)}(1) &= \beta_1 \\ y^{(2)}(0) &= \alpha_2 & y^{(2)}(1) &= \beta_2 \\ y^{(3)}(0) &= \alpha_3 \end{aligned} \right\} \quad (2.2)$$

Where f and $y(x)$ assumed real and as many times differentiable as required for $x \in [0,1]$, $\alpha_i, i = 0, 1, 2, 3$ and $\beta_i, i=0,1,2$ are finite real numbers.

The following outcomes were obtained by differentiating (2.1) n times successively:

$$y^{(7+1)}(x) = f'(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)}, y^{(6)}) = f_1(x, y, y', y'', y''', \dots, y^{(7)}) \quad (2.3)$$

$$y^{(7+2)}(x) = f''(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)}, y^{(6)}) = f_2(x, y, y', y'', y''', \dots, y^{(8)}) \quad (2.4)$$

$$y^{(7+3)}(x) = f'''(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)}, y^{(6)}) = f_3(x, y, y', y'', y''', \dots, y^{(9)}) \quad (2.5)$$

⋮

$$y^{(7+n)}(x) = f^n(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, \dots, y^{(6)}) = f_n(x, y, y', y'', y''', \dots, y^{(7+(n-1))}) \quad (2.6)$$

Assumed that:

$$\left. \begin{aligned} y^{(4)}(0) &= \delta_1 \\ y^{(5)}(0) &= \delta_2 \\ y^{(6)}(0) &= \delta_3 \end{aligned} \right\} \quad (2.7)$$

where $\delta_1, \delta_2,$ and δ_3 are unknown's parameters to be determined.

After evaluating the governing equation, equations (2.3)-(2.6) at 0, by utilizing equation (2.7), and taking into account the boundary conditions at $x=0$, the resulting set of differential equation were obtained as follows:

$$y^{(7)}(0) = f_0(x, y, y^{(1)}, y^{(2)}, y^{(3)}, \dots, y^{(6)})|_{x=0} = g_0(\delta_1, \delta_2, \delta_3) \quad (2.8)$$

$$y^{(8)}(0) = f_1(x, y, y^{(1)}, y^{(2)}, y^{(3)}, \dots, y^{(7)})|_{x=0} = g_1(\delta_1, \delta_2, \delta_3) \quad (2.9)$$

$$y^{(9)}(0) = f_2(x, y, y^{(1)}, y^{(2)}, y^{(3)}, \dots, y^{(8)})|_{x=0} = g_2(\delta_1, \delta_2, \delta_3) \quad (2.10)$$

⋮

$$y^{(7+n)}(0) = f_n(x, y, y^{(1)}, y^{(2)}, y^{(3)}, \dots, y^{(7+(n-1))})|_{x=0} = g_n(\delta_1, \delta_2, \delta_3) \quad (2.11)$$

The assumed approximate solution using the Maclaurin series expansion can be formulated as follows:

$$y(x) = y(0) + y^{(1)}(0)x + \frac{y^{(2)}(0)x^2}{2!} + \frac{y^{(3)}(0)x^3}{3!} + \dots + \frac{y^{(7+n)}(0)x^{7+n}}{(7+n)!} \quad (2.12)$$

By substituting $y(0), y^{(1)}(0), y^{(2)}(0), \dots, y^{(7+n)}(0)$ in equation (2.12), the following result was obtained as follows:

$$y(x) = \alpha_0 + \alpha_1 x + \frac{\alpha_2 x^2}{2!} + \frac{\alpha_3 x^3}{3!} + \frac{\delta_1 x^4}{4!} + \frac{\delta_2 x^5}{5!} + \frac{\delta_3 x^6}{6!} + \frac{g_0(\delta_1, \delta_2, \delta_3)x^7}{7!} + \frac{g_1(\delta_1, \delta_2, \delta_3)x^8}{8!} + \dots + \frac{g_n(\delta_1, \delta_2, \delta_3)x^{7+n}}{(7+n)!} \quad (2.13)$$

To determine the unknown's parameters $\delta_1, \delta_2, \delta_3$ the boundary conditions at $x = 1$ were used.

3.0 Numerical Results

In this section, we employ the suggested technique to solve seventh-order boundary value problems (BVPs), both linear and nonlinear. To assess the effectiveness and dependability of the method, the computed solutions are compared against both the exact solutions and with some results obtained in the exiting literature.

Example 3.1: Consider the Linear boundary value problem. (Kasi Viswanadham & Reddy, 2015; Reddy, 2016)

$$y^{(7)}(x) - xy(x) = e^x(x^2 - 2x - 6) \quad 0 \leq x \leq 1 \quad (3.1)$$

Subject to the boundary conditions

$$\begin{aligned} y(0) = 1, y^{(1)}(0) = 0, y^{(2)}(0) = -1, y^{(3)}(0) = -2 \\ y(1) = 0, y^{(1)}(1) = -e, y^{(2)}(1) = -2e \end{aligned} \quad (3.2)$$

The exact solution for the above example is given by $y(x) = (1 - x)e^x$

Assumed that

$$\left. \begin{aligned} y^{(4)}(0) &= \alpha_1 \\ y^{(5)}(0) &= \alpha_2 \\ y^{(6)}(0) &= \alpha_3 \end{aligned} \right\} \quad (3.3)$$

By successive differentiation of (3.1) and evaluating each derivative at $x = 0$ the following results were obtained.

$$\begin{aligned} y(0) = 1, y^{(1)}(0) = 0, y^{(2)}(0) = -1, y^{(3)}(0) = -2, y^{(4)}(0) = \alpha_1, y^{(5)}(0) = \alpha_2, y^{(6)}(0) = \\ \alpha_3, y^{(7)}(0) = -6, y^{(8)}(0) = -7, y^{(9)}(0) = -8, y^{(10)}(0) = -9, y^{(11)}(0) = -10, y^{(12)}(0) = (5\alpha_1 + \\ 4), y^{(13)}(0) = (6\alpha_2 + 12), \end{aligned} \quad (3.4)$$

The assumed approximate solution using the Maclaurin series expansion of equation (3.1) at $x = 0$ can be formulated as follows:

$$\begin{aligned} y(x) = y(0) + y^{(1)}(0)x + \frac{y^{(2)}(0)x^2}{2!} + \frac{y^{(3)}(0)x^3}{3!} + \frac{y^{(4)}(0)x^4}{4!} + \frac{y^{(5)}(0)x^5}{5!} + \frac{y^{(6)}(0)x^6}{6!} + \frac{y^{(7)}(0)x^7}{7!} + \frac{y^{(8)}(0)x^8}{8!} + \\ \frac{y^{(9)}(0)x^9}{9!} + \frac{y^{(10)}(0)x^{10}}{10!} + \frac{y^{(11)}(0)x^{11}}{11!} + \frac{y^{(12)}(0)x^{12}}{12!} + \frac{y^{(13)}(0)x^{13}}{13!}. \end{aligned} \quad (3.5)$$

By substituting equation (3.4) into equation (3.5) the following results was obtained.

$$\begin{aligned} y(x) = 1 + \frac{x^2}{2!} - 2\frac{x^3}{3!} + \frac{\alpha_1 x^4}{4!} + \frac{\alpha_2 x^5}{5!} + \frac{\alpha_3 x^6}{6!} - 6\frac{x^7}{7!} - 7\frac{x^8}{8!} - 8\frac{x^9}{9!} - 9\frac{x^{10}}{10!} - 10\frac{x^{11}}{11!} + (5\alpha_1 + 4)\frac{x^{12}}{12!} + \\ (6\alpha_2 + 12)\frac{x^{13}}{13!} \end{aligned} \quad (3.6)$$

The boundary condition at $x = 1$ was used in equation (3.6) then, the following system of equation was obtained follows.

$$\left. \begin{aligned} \frac{4678129}{28304640} + \frac{3991681}{95800320} \alpha_1 + \frac{8648641}{1037836800} \alpha_2 + \frac{1}{720} \alpha_3 &= 0 \\ -\frac{5348713}{2661120} + \frac{1330561}{7983360} \alpha_1 + \frac{3326401}{79833600} \alpha_2 + \frac{1}{120} \alpha_3 &= -e \\ -\frac{6789313}{2217600} + \frac{362881}{725760} \alpha_1 + \frac{1108801}{6652800} \alpha_2 + \frac{1}{24} \alpha_3 &= -2e \end{aligned} \right\} \quad (3.7)$$

After solving equation (3.7), the unknowns' parameters can be identified as

$$\alpha_1 = -3.000000177, \alpha_2 = -3.999998122, \alpha_3 = -5.000006084,$$

Hence the required approximate solution is

$$\begin{aligned} y(x) = 1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - 0.1250000074x^4 - 0.03333331768x^5 - 0.006944452894x^6 - \frac{1}{840}x^7 \\ - \frac{1}{5760}x^8 - \frac{1}{45360}x^9 - \frac{1}{403200}x^{10} - \frac{1}{3991680}x^{11} - 2.296443452 \times 10^{-8}x^{12} \\ - 1.927083451 \times 10^{-9}x^{13} \end{aligned}$$

Table 3.1: Numerical comparison between the proposed method and the other approximation techniques for example 3.1

| X | Exact Solution | Maclaurin Series Solution | Absolute Error | | |
|-----|----------------|---------------------------|---------------------------|---------------------------------|---------------|
| | | | Maclaurin Series Solution | (Kasi Viswanadham & Reddy,2015) | Reddy, (2016) |
| 0 | 1 | 1 | 0 | 0 | |
| 0.1 | 0.9946538262 | 0.9946538264 | 2.0 E-10 | 6.556511E-07 | 8.940697E-07 |
| 0.2 | 0.9771222064 | 0.9771222066 | 1.0E-10 | 9.596348E-06 | 2.145767E-06 |
| 0.3 | 0.9449011656 | 0.9449011652, | 4.0E-10 | 2.580881E-05 | 1.847744E-06 |
| 0.4 | 0.8950948188 | 0.8950948184 | 4.0E-10 | 4.059076E-05 | 2.622604E-06 |
| 0.5 | 0.8243606355 | 0.8243606351 | 4.0E-10 | 4.905462E-05 | 7.927418E-06 |
| 0.6 | 0.7288475200 | 0.7288475200 | 0 | 4.571676E-05 | 1.788139E-07 |
| 0.7 | 0.6041258121 | 0.6041258122 | 1.0E-10 | 3.087521E-05 | 2.622604E-06 |
| 0.8 | 0.4451081856 | 0.4451081856 | 0 | 1.317263E-05 | 4.947186E-06 |
| 0.9 | 0.2459603111 | 0.2459603109 | 2.0E-10 | 1.326203E-06 | 7.659197E-06 |
| 1 | 0 | -1.811771E-12 | -1.811771E-12 | | |

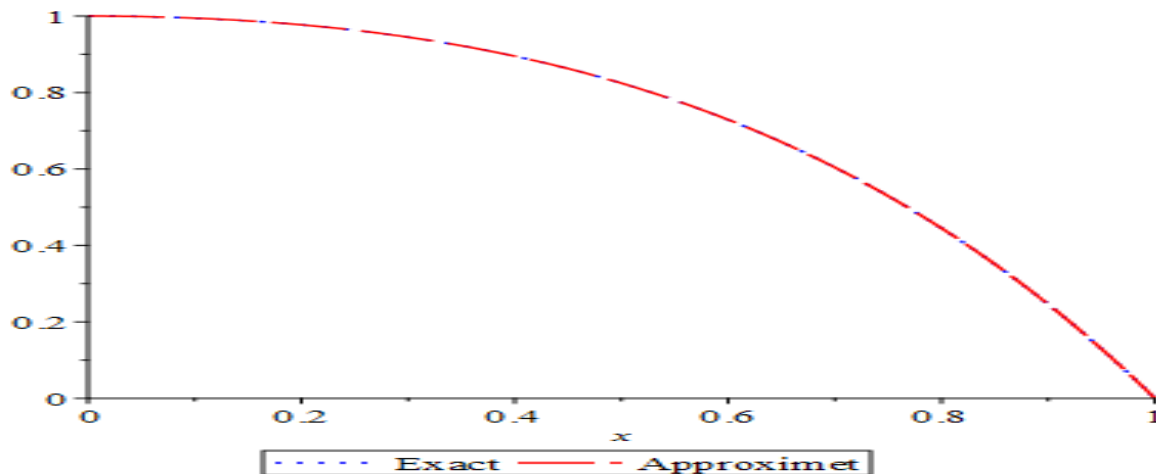


Figure 3.1: Comparison of the approximate solution with exact solution for example 3.1

Example3. 2: Consider the Linear boundary value problem. (Kasi Viswanadham & Reddy, 2015; Reddy, 2016)

$$y^{(7)}(x) + y(x) = -e^x(2x^2 + 12x + 35) \quad 0 \leq x \leq 1 \tag{3.8}$$

Subject to the boundary conditions

$$y(0) = 0, y^{(1)}(0) = 1, y^{(2)}(0) = 0, y^{(3)}(0) = -3$$

$$y(1) = 0, y^{(1)}(1) = -e, y^{(2)}(1) = -4e$$

(3.9)

The exact solution for the above example is given by $y(x) = x(1 - x)e^x$

Assumed that

$$\left. \begin{aligned} y^{(4)}(0) &= \gamma_1 \\ y^{(5)}(0) &= \gamma_2 \\ y^{(6)}(0) &= \gamma_3 \end{aligned} \right\} \quad (3.10)$$

Following as in example (3.1) the Maclaurin series approximate solution was obtained as:

$$y(x) = x - 3 \frac{x^3}{3!} + \frac{\gamma_1 x^4}{4!} + \frac{\gamma_2 x^5}{5!} + \frac{\gamma_3 x^6}{6!} - 35 \frac{x^7}{7!} - 48 \frac{x^8}{8!} - 63 \frac{x^9}{9!} - 80 \frac{x^{10}}{10!} - (\alpha_1 + 107) \frac{x^{11}}{11!} - (\alpha_2 + 135) \frac{x^{12}}{12!} - (\alpha_3 + 167) \frac{x^{13}}{13!} \quad (3.11)$$

Then, the following system of equation was obtained at a boundary point $x = 1$ from (3.11)

$$\left. \begin{aligned} \frac{1530808553}{3113510400} + \frac{1663199}{39916800} \gamma_1 + \frac{3991679}{479001600} \gamma_2 + \frac{8648639}{6227020800} \gamma_3 &= 0 \\ -\frac{268217471}{479001600} + \frac{604799}{3628800} \gamma_1 + \frac{1663199}{39916800} \gamma_2 + \frac{3991679}{479001600} \gamma_3 &= -e \\ -\frac{22440917}{6652800} + \frac{181439}{362880} \gamma_1 + \frac{604799}{3628800} \gamma_2 + \frac{1663199}{39916800} \gamma_3 &= -4e \end{aligned} \right\} \quad (3.12)$$

After solving equation (3.12), the unknowns' parameters can be identified as:

$$\gamma_1 = -8.000001840, \gamma_2 = -14.99997962, \gamma_3 = -24.00006852$$

Hence, the required Maclaurin Series solution is

$$y(x) = x - \frac{1}{2} x^3 - 0.3333334100 x^4 - 0.1249998302 x^5 - 0.03333342850 x^6 - \frac{1}{144} x^7 - \frac{1}{840} x^8 - \frac{1}{5760} x^9 - \frac{1}{45360} x^{10} - 0.000002480158684 x^{11} - 2.505211264 \times 10^{-7} x^{12} - 2.296442169 \times 10^{-8} x^{13}$$

Table 3.2: Numerical comparison between the proposed method and the other approximation techniques for example 3.2

| x | Exact Solution | Maclaurin series solution | Absolute Error | | |
|-----|----------------|---------------------------|---------------------------|---------------------------------|---------------|
| | | | Maclaurin Series Solution | (Kasi Viswanadham & Reddy,2015) | Reddy, (2016) |
| 0 | 0 | 0 | 0 | - | |
| 0.1 | 0.09946538262 | 0.09946538263 | 1.0 E-11 | 1.415610E-07 | 3.054738E-07 |
| 0.2 | 0.1954244413 | 0.1954244413 | 0 | 6.407499E-07 | 5.81453E-07 |
| 0.3 | 0.2834703497 | 0.2834703494 | 3.0E-10 | 2.920628E-06 | 2.592802E-06 |
| 0.4 | 0.3580379275 | 0.3580379268 | 7.0E-10 | 4.410744E-06 | 8.940697E-07 |
| 0.5 | 0.4121803178 | 0.4121803167 | 1.0E-09 | 6.735325E-06 | 3.218651E-06 |
| 0.6 | 0.4373085120 | 0.4373085111 | 9.0 E-10 | 6.407499E-06 | 1.877546E-06 |
| 0.7 | 0.4228880685 | 0.4228880676 | 9.0E-10 | 3.665686E-06 | 2.950430E-06 |
| 0.8 | 0.3560865485 | 0.3560865478 | 7.E-10 | 3.278255E-07 | 5.55802E-06 |
| 0.9 | 0.2213642800 | 0.2213642799 | 1.0E-10 | 1.430511E-06 | 8.821487E-06 |
| 1 | 0 | 5.528791E-11 | 5.528791E-11 | | |

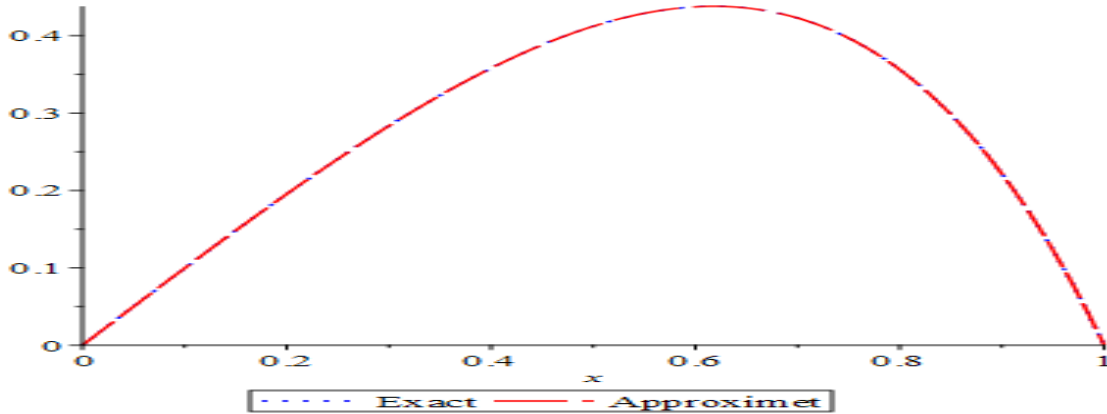


Figure 3.2: Comparison of the approximate solution with exact solution for example 3.2

Example3. 3: Consider the Nonlinear boundary value problem.

$$y^{(7)}(x) = e^{-x}(y(x))^2 \quad 0 < x < 1 \tag{3.13}$$

Subject to the boundary conditions

$$\begin{aligned} y(0) = 1, y^{(1)}(0) = 1, y^{(2)}(0) = 1, y^{(3)}(0) = 1 \\ y(1) = e, y^{(1)}(1) = e, y^{(2)}(1) = e \end{aligned} \tag{3.14}$$

The exact solution for the above example is given by $y(x) = e^x$

Assumed that

$$\left. \begin{aligned} y^{(4)}(0) &= \delta_1 \\ y^{(5)}(0) &= \delta_2 \\ y^{(6)}(0) &= \delta_3 \end{aligned} \right\} \tag{3.15}$$

Following as in example (3.1), the Maclaurin series approximate solution can be written as follows:

$$y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{\delta_1 x^4}{4!} + \frac{\delta_2 x^5}{5!} + \frac{\delta_3 x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \frac{x^{10}}{10!} + (2\alpha_1 - 1) \frac{x^{11}}{11!} + (2\alpha_2 - 1) \frac{x^{12}}{12!} + (2\alpha_3 - 1) \frac{x^{13}}{13!} \tag{3.16}$$

Then, the following system of equation was obtained at a boundary point $x = 1$ from (3.16)

$$\left. \begin{aligned} \frac{1186199819}{444787200} + \frac{831601}{19958400} \delta_1 + \frac{1995841}{239500800} \delta_2 + \frac{4324321}{3113510400} \delta_3 &= e \\ \frac{47931095}{19160064} + \frac{302401}{1814400} \delta_1 + \frac{831601}{19958400} \delta_2 + \frac{1995841}{239500800} \delta_3 &= e \\ \frac{20057617}{9979200} + \frac{90721}{181440} \delta_1 + \frac{302401}{1814400} \delta_2 + \frac{831601}{19958400} \delta_3 &= e \end{aligned} \right\} \tag{3.17}$$

After solving equation (3.17), the unknowns' parameters can be identified as:

$$\delta_1 = .9999998502, \delta_2 = 1.000001148, \quad \delta_3 = 0.9999972497$$

Hence, the required Maclaurin Series solution is

$$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + 0.04166666042x^4 + 0.008333342900x^5 + 0.001388885069x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 + \frac{1}{362880}x^9 + \frac{1}{3628800}x^{10} + 2.505210087 \times 10^{-8}x^{11} + 2.087680492 \times 10^{-9}x^{12} + 1.605895550 \times 10^{-10}x^{13}$$

Table 3.3: Numerical comparison between the proposed method and the other approximation techniques for example 3.3

| x | Exact Solution | Maclaurin series solution | Absolute Error | | |
|-----|----------------|---------------------------|---------------------------|------------------------------|-----------------------------------|
| | | | Maclaurin series solution | Siddiqi and Iftikhar. (2015) | Al- Mamun And Asaduzzaman, (2019) |
| 0 | 1 | 1 | 0 | 1.32455E-09 | 0 |
| 0.1 | 1.105170918 | 1.105170918 | 0 | 5.26137E-10 | 7.65E-12 |
| 0.2 | 1.221402758 | 1.221402759 | 1.0E-9 | 1.64015E-09 | 9.31E-11 |
| 0.3 | 1.349858808 | 1.349858807 | 1.0E-9 | 4.56139E-09 | 3.43E-10 |
| 0.4 | 1.491824698 | 1.491824698 | 0 | 2.9619E-09 | 7.48E-10 |
| 0.5 | 1.648721271 | 1.648721269 | 2.0E-9 | 7.54889E-10 | 1.16E-9 |
| 0.6 | 1.822118800 | 1.822118801 | 1.0E-9 | 2.67612E-09 | 1.36E-9 |
| 0.7 | 2.013752707 | 2.013752707 | 0 | 8.42306E-10 | 1.19E-9 |
| 0.8 | 2.225540928 | 2.225540928 | 0 | 1.16866E-09 | 6.76E-10 |
| 0.9 | 2.459603111 | 2.459603111 | 0 | 4.4716E-09 | 1.55E-10 |
| 1 | 2.718281828 | 2.718281829 | 1.0E-9 | 1.02746E-09 | 1.21E-11 |

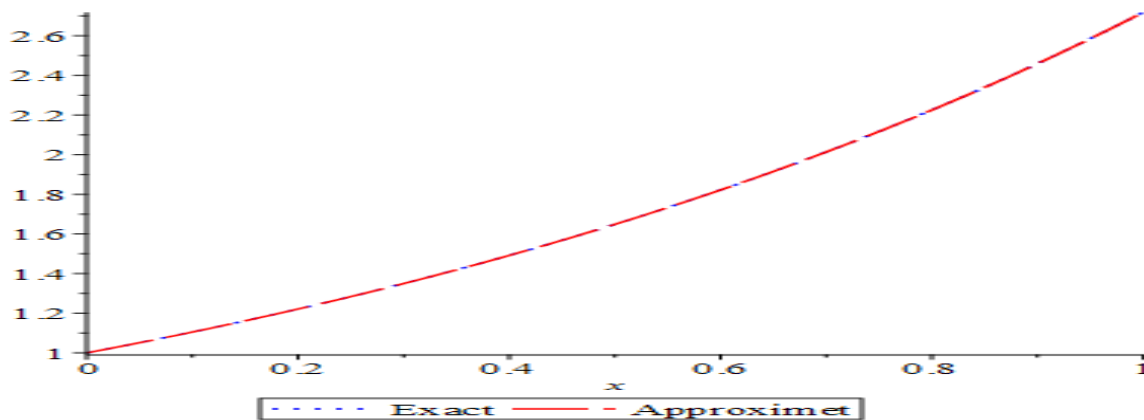


Figure 3.3: Comparison of the approximate solution with exact solution for example 3.3

It's evident that the numerical results in tables 3.1 to 3.4 align well with the exact solution and with some established techniques in the literature. The agreement between the numerical and exact solutions, as shown in figures 3.1 to 3.4, further reinforces the effectiveness of the suggested approach.

4.0 CONCLUSION

The study demonstrates the successful application of the Maclaurin series method to solve linear and

nonlinear seventh order boundary value problems (BVPs). The method's efficiency and reliability are confirmed through the solution of three examples, which are compared to exact solutions and other existing techniques in the literature. It shows that proposed method gives better results than those obtained from other techniques in the literature such as quintic B-splines, finite element method, and variational iteration method, as indicated by tabulated data.

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