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An Implicit Two-Step Hybrid Block Method based on Chebyshev Polynomial for Solving First Order Initial Value Problems in Ordinary Differential Equations

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ABSTRACT

This research focuses on the derivation of new implicit two step block hybrid method for the solution of first order IVPs in ODEs. The new methods s derived based on multistep collocation using Chebyshev polynomials as a bases functions at some selected points to get a continuous linear multistep method. The continuous methods are evaluated at some off-grid points to generate the discrete schemes for step number $k = 2$ which conveniently constitutes the block method. Basic properties of the developed method is examined and the method is found to be zero stable, consistent, convergent and of uniform order 6. The efficiency of the method is tested on some numerical examples in the literature. On comparison, the method developed performed favorably when compared with the existing methods. Therefore, the method is recommended for the solution of general first order initial value problems in ordinary differential equations.

Keywords: Differential Equation, Initial Value Problems, Linear multistep method, Chebyshev Polynomial, Interpolation, Collocation

1. INTRODUCTION

A broad variety of classical techniques have since been developed for finding approximate solutions to ordinary differential equation (ODEs). Many researchers have developed different methods for solving first order ordinary differential equations among which are A. Abdullahi, J. P. Chollom, & Y. A. Yahaya, (2014), R. B. Adeniyi, & M. O. Alabi, (2007), Ehigie, J. O., Okunuga, S. A., & Sololuwa, A. B. (2011), multi-step collocation methods of M. T. Chu, and H. Hamilton, (1987) just to mention but a few. In this research work, we consider the numerical computational methods for first ODEs of the form

$$y' = f(x, y), \quad y(a) = y_0, \quad x \in (a, b) \quad (1)$$

where x_0 is the initial point y_0 is the solution at the initial point and f is assumed to be continuous

and satisfy Lipschitz's condition for the existence and uniqueness of solution (Fatunla, 1980). Linear multistep methods (LMMs) are very prevalent for solving initial value problems (IVPs) of ordinary differential equations (ODEs). They are also applicable to solving higher order ODEs. Generally, LMMs are not self-starting especially when step number is greater than one hence, need starting values from single step methods and Runge-Kutta family of methods (Jator and Li (2009).

In this research, we intend to combine the qualities of hybrid methods, block methods and approximate solution using Chebyshev polynomial as a trial function to derive a new method for the solution of first ODEs are utilized.

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solution using Chebyshev polynomial as a trial function to derive a new method for the solution of first ODEs are utilized.

2. Derivation of the Method

The approach adopted in this section entails substituting into (1.1) an approximate solution of the form

$$y(x) = \sum_{j=0}^{p+q-1} \alpha_j T_j(x) \quad (2)$$

where α_j is unknown coefficients and $T_j(x)$ are polynomial basis functions of degree $p+q-1$ in a manner that $1 \leq p \leq k$ and $q > 0$. The integer $k \geq 1$ denote the step number of the method.

$T_j(x)$ is the Chebyshev polynomial generated by the formula:

$$T_j(x) = \frac{(-2)^j}{(2j)!} j! \sqrt{1-x^2} \frac{d^j}{dx^j} (1-x^2)^{j-\frac{1}{2}} \quad (3)$$

For sake of reporting, we present some few terms of Chebyshev polynomial as

$$T_0(x) = 1,$$

$$T_1(x) = x,$$

$$T_2(x) = 2x^2 - 1,$$

$$T_3(x) = 4x^3 - 3x,$$

$$T_4(x) = 8x^4 - 8x^2 + 1,$$

$$T_5(x) = 16x^5 - 20x^3 + 5x,$$

$$T_6(x) = 32x^6 - 48x^4 + 8x^2 - 1,$$

$$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x,$$

$$T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1,$$

From (2)

$$y'(x) = \sum_{j=0}^{p+q-1} \alpha_j T'_j(x) \quad (4)$$

Putting (4) into (1) we obtained

$$f(x, y) = \sum_{j=0}^{p+q-1} \alpha_j T'_j(x) \quad (5)$$

To derive the method, some off step points are carefully introduced to guarantee zero stability.

3. Specification of the Method

To derive this method, two off-step points and one non-interpolated and non-collocation point is introduced. This off step points are carefully selected in order to guarantee zero stability condition. For this method, the off step points are $\left\{ \frac{1}{2}, \frac{3}{2} \text{ and } \frac{7}{8} \right\}$. Using (3.1), we have a polynomial of degree $p+q-1$ as follows:

$$y(x) = \sum_{j=0}^6 \alpha_j T_j(x) \quad (6)$$

with first derivative given as

$$f(x, y) = \sum_{j=0}^6 \alpha_j T'_j(x) \quad (7)$$

Now interpolating at $x_{n+p}, p=0, p=\frac{1}{2}$ and

collocating at $x_{n+q}, q=0, \frac{1}{2}, 1, \frac{3}{2}, 2$ leads to the system of equations as $AX=U$

using the Gaussian elimination method to solve (8) gives the unknown coefficient

$\alpha_i, \beta_i, \left(i=0, \frac{1}{2}, \frac{3}{2}, 1, 2 \right)$ which are then substituted into equation (6) and simplified to give the continuous hybrid method of the form;

$$y(x) = \alpha_0(x)y_n + \alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}} + h \left[\sum_{j=0}^2 \beta_j(x)f_{n+j} + \beta_{\frac{1}{2}}(x)f_{n+\frac{1}{2}} + \beta_{\frac{3}{2}}(x)f_{n+\frac{3}{2}} \right]$$

$$\begin{bmatrix} 9 & 22x_n & 69x_n^2 & -100x_n^3 & -245x_n^4 & 126x_n^5 & 231x_n^6 \\ 9 & 22x_{n+\frac{1}{2}} & 69x_{n+\frac{1}{2}}^2 & -100x_{n+\frac{1}{2}}^3 & -245x_{n+\frac{1}{2}}^4 & 126x_{n+\frac{1}{2}}^5 & 231x_{n+\frac{1}{2}}^6 \\ 0 & 22 & 138x_n & -300x_n^2 & -980x_n^3 & 630x_n^4 & 1386x_n^5 \\ 0 & 22 & 138x_{n+\frac{1}{2}} & -300x_{n+\frac{1}{2}}^2 & -980x_{n+\frac{1}{2}}^3 & 630x_{n+\frac{1}{2}}^4 & 1386x_{n+\frac{1}{2}}^5 \\ 0 & 22 & 138x_{n+\frac{3}{2}} & -300x_{n+\frac{3}{2}}^2 & -980x_{n+\frac{3}{2}}^3 & 630x_{n+\frac{3}{2}}^4 & 1386x_{n+\frac{3}{2}}^5 \\ 0 & 22 & 138x_{n+1} & -300x_{n+1}^2 & -980x_{n+1}^3 & 630x_{n+1}^4 & 1386x_{n+1}^5 \\ 0 & 22 & 138x_{n+2} & -300x_{n+2}^2 & -980x_{n+2}^3 & 630x_{n+2}^4 & 1386x_{n+2}^5 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_{\frac{1}{2}} \\ \beta_0 \\ \beta_{\frac{1}{2}} \\ \beta_{\frac{3}{2}} \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+\frac{1}{2}} \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{2}} \\ f_{n+1} \\ f_{n+2} \end{bmatrix} \tag{9}$$

where

$$\alpha_0 = 1 - \frac{64}{3} \frac{(-x_n+x)^2}{h^2} + \frac{1600}{27} \frac{(-x_n+x)^3}{h^3} - \frac{560}{9} \frac{(-x_n+x)^4}{h^4} + \frac{256}{9} \frac{(-x_n+x)^5}{h^5} - \frac{128}{27} \frac{(-x_n+x)^6}{h^6};$$

$$\alpha_{\frac{1}{2}} = \frac{64}{3} \frac{(-x_n+x)^2}{h^2} - \frac{1600}{27} \frac{(-x_n+x)^3}{h^3} + \frac{560}{9} \frac{(-x_n+x)^4}{h^4} - \frac{256}{9} \frac{(-x_n+x)^5}{h^5} + \frac{128}{27} \frac{(-x_n+x)^6}{h^6};$$

$$\beta_0 = -x_n + x - \frac{3133}{540} \frac{(-x_n+x)^2}{h} + \frac{5965}{486} \frac{(-x_n+x)^3}{h^2} - \frac{946}{81} \frac{(-x_n+x)^4}{h^3} + \frac{2062}{405} \frac{(-x_n+x)^5}{h^4} - \frac{1004}{1215} \frac{(-x_n+x)^6}{h^5};$$

$$\beta_{\frac{1}{2}} = -\frac{752}{135} \frac{(-x_n+x)^2}{h} + \frac{5056}{243} \frac{(-x_n+x)^3}{h^2} - \frac{2018}{81} \frac{(-x_n+x)^4}{h^3} + \frac{4952}{405} \frac{(-x_n+x)^5}{h^4} - \frac{2584}{1215} \frac{(-x_n+x)^6}{h^5};$$

$$\beta_1 = \frac{41}{45} \frac{(-x_n+x)^2}{h} - \frac{367}{81} \frac{(-x_n+x)^3}{h^2} + \frac{200}{27} \frac{(-x_n+x)^4}{h^3} - \frac{596}{135} \frac{(-x_n+x)^5}{h^4} + \frac{352}{405} \frac{(-x_n+x)^6}{h^5};$$

$$\begin{aligned} \beta_{\frac{3}{2}} &= \frac{32}{135} \frac{(-x_n+x)^2}{h} + \frac{304}{243} \frac{(-x_n+x)^3}{h^2} - \frac{182}{81} \frac{(-x_n+x)^4}{h^3} + \frac{632}{405} \frac{(-x_n+x)^5}{h^4} \\ &\quad - \frac{424}{1215} \frac{(-x_n+x)^6}{h^5}; \\ \beta_2 &= \frac{17}{540} \frac{(-x_n+x)^2}{h} - \frac{83}{486} \frac{(-x_n+x)^3}{h^2} + \frac{26}{81} \frac{(-x_n+x)^4}{h^3} - \frac{98}{405} \frac{(-x_n+x)^5}{h^4} \\ &\quad + \frac{76}{1215} \frac{(-x_n+x)^6}{h^5}; \end{aligned} \tag{10}$$

Now, evaluating (9) $\frac{3}{2}, 1, \frac{7}{8}, 2$ and its first derivative at $x_{n+\frac{7}{8}}$ we obtain the following five discrete schemes which constitute the block form.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{7}{8}} \\ y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [y_n] + \begin{bmatrix} \frac{214067h}{1474560} \\ 1487h \\ 10080 \\ \frac{61h}{420} \\ \frac{159h}{1120} \\ \frac{7h}{45} \end{bmatrix} [f_n] \\ + \begin{bmatrix} -\frac{63h}{80} & \frac{309043h}{368640} & \frac{181447h}{245760} & -\frac{24353h}{368640} & \frac{9947h}{1474560} \\ -\frac{128h}{105} & \frac{503h}{720} & \frac{113h}{120} & -\frac{11h}{144} & \frac{11h}{1440} \\ -\frac{2048h}{2835} & \frac{113h}{135} & \frac{4h}{5} & -\frac{h}{15} & \frac{11h}{1620} \\ -\frac{128h}{105} & \frac{71h}{80} & \frac{63h}{40} & \frac{9h}{80} & \frac{h}{480} \\ 0 & \frac{32h}{45} & \frac{4h}{15} & \frac{32h}{45} & \frac{7h}{45} \end{bmatrix} \begin{bmatrix} f_{n+\frac{7}{8}} \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} \tag{11}$$

This five discrete schemes are shown as

$$\begin{aligned} y_{n+1} &= \frac{11}{27} y_n + \frac{16}{27} y_{n+\frac{1}{2}} + \frac{281}{4860} h f_n + \frac{514}{1215} h f_{n+\frac{1}{2}} + \frac{98}{405} h f_{n+1} \\ &\quad - \frac{26}{1215} h f_{n+\frac{3}{2}} + \frac{11}{4860} h f_{n+2}; \end{aligned}$$

$$y_{n+\frac{3}{2}} = y_{n+\frac{1}{2}} - \frac{1}{180}hf_n + \frac{17}{90}hf_{n+\frac{1}{2}} + \frac{19}{30}hf_{n+1} + \frac{17}{90}hf_{n+\frac{3}{2}} - \frac{1}{180}hf_{n+2};$$

$$y_{n+2} = y_n + \frac{7}{45}hf_n + \frac{32}{45}hf_{n+\frac{1}{2}} + \frac{4}{15}hf_{n+1} + \frac{32}{45}hf_{n+\frac{3}{2}} + \frac{7}{45}hf_{n+2};$$

$$y_{n+\frac{7}{8}} = \frac{725}{2048}y_n + \frac{1323}{2048}y_{n+\frac{1}{2}} + \frac{147091}{2949120}hf_n + \frac{570703}{1474560}hf_{n+\frac{1}{2}} + \frac{7987}{61440}hf_{n+1}$$

$$- \frac{24647}{1474560}hf_{n+\frac{3}{2}} + \frac{5341}{2949120}hf_{n+2};$$

$$\frac{105}{128}y_{n+\frac{1}{2}} = \frac{105}{128}y_n + \frac{1487}{12288}hf_n - hf_{n+\frac{7}{8}} + \frac{3521}{6144}hf_{n+\frac{1}{2}} + \frac{791}{1024}hf_{n+1}$$

$$- \frac{385}{6144}hf_{n+\frac{3}{2}} + \frac{77}{12288}hf_{n+2};$$

(12)

4. Analysis of Method

4.1 Order and error constant of the new method

The order and error constant of the new method is obtain, the equation can be written in Taylor series expansion about the point x , where the constant coefficients $C_q, q=0,1,2,\dots$ are given in the form.

Expressing equation (12) in the form of

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{7}{8}} \\ y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [y_n] + \begin{bmatrix} \frac{214067h}{1474560} \\ \frac{1487h}{10080} \\ \frac{61h}{420} \\ \frac{159h}{1120} \\ \frac{7h}{45} \end{bmatrix} [f_n]$$

$$+ \begin{bmatrix} -\frac{63h}{80} & \frac{309043h}{368640} & \frac{181447h}{245760} & -\frac{24353h}{368640} & \frac{9947h}{1474560} \\ -\frac{128h}{105} & \frac{503h}{720} & \frac{113h}{120} & -\frac{11h}{144} & \frac{11h}{1440} \\ -\frac{2048h}{2835} & \frac{113h}{135} & \frac{4h}{5} & -\frac{h}{15} & \frac{11h}{1620} \\ -\frac{128h}{105} & \frac{71h}{80} & \frac{63h}{40} & \frac{9h}{80} & \frac{h}{480} \\ 0 & \frac{32h}{45} & \frac{4h}{15} & \frac{32h}{45} & \frac{7h}{45} \end{bmatrix} \begin{bmatrix} f_{n+\frac{7}{8}} \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix}$$

(13)

Expanding (13) in Taylor series about x_n and comparing the coefficient of h gives

$$\left[\sum_{j=0}^{\infty} \frac{\left(\frac{7}{8}h\right)^j}{j!} y_{n+\frac{7}{8}}^{(j)} - y_n - \frac{214067h}{1474560} \sum_{j=0}^{\infty} \frac{(h)^{j+1}}{j!} y_n^{(j+1)} \left[-\frac{63h}{80} \left(\frac{7}{8}\right)^j + \frac{309043h}{368640} \left(\frac{1}{2}\right)^j + \frac{181447h}{245760} \left(\frac{3}{2}\right)^j + \frac{9947h}{1474560} \left(2\right)^j \right] \right. \\ \left. \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}h\right)^j}{j!} y_{n+\frac{1}{2}}^{(j)} - y_n - \frac{1487h}{10080} \sum_{j=0}^{\infty} \frac{(h)^{j+1}}{j!} y_n^{(j+1)} \left[-\frac{128h}{105} \left(\frac{7}{8}\right)^j + \frac{503h}{720} \left(\frac{1}{2}\right)^j + \frac{113h}{120} \left(\frac{3}{2}\right)^j + \frac{11h}{1440} \left(2\right)^j \right] \right. \\ \left. \sum_{j=0}^{\infty} \frac{(h)^j}{j!} y_{n+1}^{(j)} - y_n - \frac{61h}{420} \sum_{j=0}^{\infty} \frac{(h)^{j+1}}{j!} y_n^{(j+1)} \left[-\frac{2048h}{2835} \left(\frac{7}{8}\right)^j + \frac{113h}{135} \left(\frac{1}{2}\right)^j + \frac{71h}{80} \left(\frac{3}{2}\right)^j + \frac{63h}{40} \left(2\right)^j \right] \right. \\ \left. \sum_{j=0}^{\infty} \frac{\left(\frac{3}{2}h\right)^j}{j!} y_{n+\frac{3}{2}}^{(j)} - y_n - \frac{159h}{1120} \sum_{j=0}^{\infty} \frac{(h)^{j+1}}{j!} y_n^{(j+1)} \left[-\frac{128h}{105} \left(\frac{7}{8}\right)^j + \frac{71h}{80} \left(\frac{1}{2}\right)^j + \frac{9h}{80} \left(\frac{3}{2}\right)^j + \frac{h}{480} \left(2\right)^j \right] \right. \\ \left. \sum_{j=0}^{\infty} \frac{(2h)^j}{j!} y_{n+2}^{(j)} - y_n - \frac{7h}{45} \sum_{j=0}^{\infty} \frac{(h)^{j+1}}{j!} y_n^{(j+1)} \left[\frac{32h}{45} \left(\frac{1}{2}\right)^j + \frac{4h}{15} \left(1\right)^j + \frac{32h}{45} \left(\frac{3}{2}\right)^j + \frac{7h}{45} \left(2\right)^j \right] \right]$$

(14)

$$C_0 = \alpha_0 + \alpha_{\frac{7}{8}} + \alpha_{\frac{1}{2}} + \alpha_1 + \alpha_{\frac{3}{2}} + \alpha_2$$

$$C_2 = \frac{1}{2!} \left(\left(\frac{7}{8}\right)^2 \alpha_{\frac{7}{8}} + \left(\frac{1}{2}\right)^2 \alpha_{\frac{1}{2}} + \alpha_1 + \left(\frac{3}{2}\right)^2 \alpha_{\frac{3}{2}} + (2)^2 \alpha_2 \right) - \left(\beta_{\frac{7}{8}} + \beta_{\frac{1}{2}} + \beta_1 + \beta_{\frac{3}{2}} + \beta_2 \right)$$

$$C_2 = \frac{1}{2!} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (2)^2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) \\ - \left(\begin{pmatrix} -\frac{63}{80} \\ -\frac{128}{105} \\ -\frac{2048}{2835} \\ -\frac{128}{105} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{309043}{368640} \\ \frac{503}{720} \\ \frac{113}{135} \\ \frac{71}{80} \\ \frac{32}{45} \end{pmatrix} + \begin{pmatrix} \frac{181447}{245760} \\ \frac{113}{120} \\ \frac{4}{5} \\ \frac{63}{40} \\ \frac{4}{15} \end{pmatrix} + \begin{pmatrix} -\frac{24353}{368640} \\ -\frac{11}{144} \\ -\frac{1}{15} \\ \frac{9}{80} \\ \frac{32}{45} \end{pmatrix} \right) \\ + (2) \begin{pmatrix} \frac{9947}{1474560} \\ \frac{11}{1440} \\ \frac{11}{1620} \\ \frac{1}{480} \\ \frac{7}{45} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_0 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_3 = \frac{1}{3!} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (2)^3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$- \frac{1}{2!} \left(\begin{bmatrix} -\frac{63}{80} \\ -\frac{128}{105} \\ -\frac{2048}{2835} \\ -\frac{128}{105} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{309043}{368640} \\ \frac{503}{720} \\ \frac{113}{135} \\ \frac{71}{80} \\ \frac{32}{45} \end{bmatrix} + \begin{bmatrix} \frac{181447}{245760} \\ \frac{113}{120} \\ \frac{4}{5} \\ \frac{63}{40} \\ \frac{4}{15} \end{bmatrix} \right)$$

$$+ \begin{bmatrix} -\frac{24353}{368640} \\ -\frac{11}{144} \\ -\frac{1}{15} \\ \frac{9}{80} \\ \frac{32}{45} \end{bmatrix} + (2)^2 \begin{bmatrix} \frac{9947}{1474560} \\ \frac{11}{1440} \\ \frac{11}{1620} \\ \frac{1}{480} \\ \frac{7}{45} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_4 = \frac{1}{4!} \left(\left(\frac{7}{8} \right)^4 \alpha_{\frac{7}{8}} + \left(\frac{1}{2} \right)^4 \alpha_{\frac{1}{2}} + \alpha_1 + \left(\frac{3}{2} \right)^4 \alpha_{\frac{3}{2}} + (2)^4 \alpha_2 \right) - \frac{1}{3!} \left(\left(\frac{7}{8} \right)^3 \beta_{\frac{7}{8}} \right.$$

$$\left. + \left(\frac{1}{2} \right)^3 \beta_{\frac{1}{2}} + \beta_1 + \left(\frac{3}{2} \right)^3 \beta_{\frac{3}{2}} + (2)^3 \beta_2 \right):$$

$$C_4 = \frac{1}{4!} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (2)^4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$- \frac{1}{3!} \left(\begin{bmatrix} -\frac{63}{80} \\ -\frac{128}{105} \\ -\frac{2048}{2835} \\ -\frac{128}{105} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{309043}{368640} \\ \frac{503}{720} \\ \frac{113}{135} \\ \frac{71}{80} \\ \frac{32}{45} \end{bmatrix} + \begin{bmatrix} \frac{181447}{245760} \\ \frac{113}{120} \\ \frac{4}{5} \\ \frac{63}{40} \\ \frac{4}{15} \end{bmatrix} \right)$$

$$+ \begin{bmatrix} -\frac{24353}{368640} \\ -\frac{11}{144} \\ -\frac{1}{15} \\ \frac{9}{80} \\ \frac{32}{45} \end{bmatrix} + (2)^3 \begin{bmatrix} \frac{9947}{1474560} \\ \frac{11}{1440} \\ \frac{11}{1620} \\ \frac{1}{480} \\ \frac{7}{45} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_5 = \frac{1}{5!} \left(\left(\frac{7}{8} \right)^5 \alpha_{\frac{7}{8}} + \left(\frac{1}{2} \right)^5 \alpha_{\frac{1}{2}} + \alpha_1 + \left(\frac{3}{2} \right)^5 \alpha_{\frac{3}{2}} + (2)^5 \alpha_2 \right) - \frac{1}{4!} \left(\left(\frac{7}{8} \right)^4 \beta_{\frac{7}{8}} \right.$$

$$\left. + \left(\frac{1}{2} \right)^4 \beta_{\frac{1}{2}} + \beta_1 + \left(\frac{3}{2} \right)^4 \beta_{\frac{3}{2}} + (2)^4 \beta_2 \right):$$

$$C_5 = \frac{1}{5!} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \left(\frac{1}{2}\right)^5 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \left(\frac{3}{2}\right)^5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + (2)^5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$- \frac{1}{4!} \begin{pmatrix} -\frac{63}{80} \\ -\frac{128}{105} \\ -\frac{2048}{2835} \\ -\frac{128}{105} \\ 0 \end{pmatrix} + \left(\frac{1}{2}\right)^4 \begin{pmatrix} \frac{309043}{368640} \\ \frac{503}{720} \\ \frac{113}{135} \\ \frac{71}{80} \\ \frac{32}{45} \end{pmatrix} + \begin{pmatrix} \frac{181447}{245760} \\ \frac{113}{120} \\ \frac{4}{5} \\ \frac{63}{40} \\ \frac{4}{15} \end{pmatrix}$$

$$+ \left(\frac{3}{2}\right)^4 \begin{pmatrix} -\frac{24353}{368640} \\ -\frac{11}{144} \\ -\frac{1}{15} \\ \frac{9}{80} \\ \frac{32}{45} \end{pmatrix} + (2)^4 \begin{pmatrix} \frac{9947}{1474560} \\ \frac{11}{1440} \\ \frac{11}{1620} \\ \frac{1}{480} \\ \frac{7}{45} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_6 = \frac{1}{6!} \left(\left(\frac{7}{8}\right)^6 \alpha_{\frac{7}{8}} + \left(\frac{1}{2}\right)^6 \alpha_{\frac{1}{2}} + \alpha_1 + \left(\frac{3}{2}\right)^6 \alpha_{\frac{3}{2}} + (2)^6 \alpha_2 \right) - \frac{1}{5!} \left(\left(\frac{7}{8}\right)^5 \beta_{\frac{7}{8}} + \left(\frac{1}{2}\right)^5 \beta_{\frac{1}{2}} + \beta_1 + \left(\frac{3}{2}\right)^5 \beta_{\frac{3}{2}} + (2)^5 \beta_2 \right)$$

$$C_6 = \frac{1}{6!} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \left(\frac{1}{2}\right)^6 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \left(\frac{3}{2}\right)^6 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + (2)^6 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$- \frac{1}{5!} \begin{pmatrix} -\frac{63}{80} \\ -\frac{128}{105} \\ -\frac{2048}{2835} \\ -\frac{128}{105} \\ 0 \end{pmatrix} + \left(\frac{1}{2}\right)^5 \begin{pmatrix} \frac{309043}{368640} \\ \frac{503}{720} \\ \frac{113}{135} \\ \frac{71}{80} \\ \frac{32}{45} \end{pmatrix} + \begin{pmatrix} \frac{181447}{245760} \\ \frac{113}{120} \\ \frac{4}{5} \\ \frac{63}{40} \\ \frac{4}{15} \end{pmatrix}$$

$$+ \left(\frac{3}{2}\right)^5 \begin{pmatrix} -\frac{24353}{368640} \\ -\frac{11}{144} \\ -\frac{1}{15} \\ \frac{9}{80} \\ \frac{32}{45} \end{pmatrix} + (2)^5 \begin{pmatrix} \frac{9947}{1474560} \\ \frac{11}{1440} \\ \frac{11}{1620} \\ \frac{1}{480} \\ \frac{7}{45} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_7 = \frac{1}{7!} \left(\left(\frac{7}{8}\right)^7 \alpha_{\frac{7}{8}} + \left(\frac{1}{2}\right)^7 \alpha_{\frac{1}{2}} + \alpha_1 + \left(\frac{3}{2}\right)^7 \alpha_{\frac{3}{2}} + (2)^7 \alpha_2 \right) - \frac{1}{6!} \left(\left(\frac{7}{8}\right)^6 \beta_{\frac{7}{8}} + \left(\frac{1}{2}\right)^6 \beta_{\frac{1}{2}} + \beta_1 + \left(\frac{3}{2}\right)^6 \beta_{\frac{3}{2}} + (2)^6 \beta_2 \right)$$

The truncation error is then given as $C_{p+1}=7$ in which $p=6$, comparing the coefficient of h gives $C_0=C_1=C_2=C_3=\dots=C_6=0$ and

$$C_7 = \left[-\frac{53165}{1811939328}, -\frac{199}{6193152}, -\frac{19}{645120}, -\frac{5}{229376}, -\frac{1}{15120} \right]^T$$

Therefore, the truncation error is then given as

$$C_{p+1}=7 \text{ in which } p=[6, 6, 6, 6, 6]^T,$$

Therefore, the new method (12) is of uniform order six (6) with error constant

$$C_7 = \left[-\frac{53165}{1811939328}, -\frac{199}{6193152}, -\frac{19}{645120}, -\frac{5}{229376}, -\frac{1}{15120} \right]^T$$

4.2 Consistency

The hybrid block method is said to be consistent if it has an order more than or equal to one. Therefore, our method is consistent, since it is of order six (6). (Lambert, 1973)

4.3 Zero Stability of Method

The implicit hybrid method (12) for k=2, with three off grid collocation points expressed in the form

$$\rho(z) = \det \left[\sum_{j=0}^k A^{(i)} z^{k-i} \right] = 0 \text{ gives}$$

$$\rho(z) = z \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = z^3(z-1)^2$$

$\rho(z) = z^3(z-1)^2 = 0, z = 0, 0, 0, 1, 1$
Hence, our method (12) is zero-stable.

4.4 Convergence

Theorem 4.1 (Henrici, 1962).
Zero stability and consistency are sufficient conditions for a linear multistep method to be convergent. Since our new hybrid method (12) is consistent and zero-stable, by Henrici, 1962 the hybrid method (12) is convergent.

4.5 Region of Absolute Stability

To plot the region of absolute stability of the new method (12), the methods were formulated as general linear method and the stability polynomial for method (12) is computed to give

$$\left(-\frac{3}{1280} w^4 - \frac{7}{3840} w^5 \right) h^5 + \left(-\frac{91}{3840} w^4 + \frac{223}{11520} w^5 \right) h^4$$

$$+ \left(-\frac{103}{768} w^4 - \frac{89}{768} w^5 \right) h^3 + \left(-\frac{23}{48} w^4 + \frac{7}{16} w^5 \right) h^2$$

$$+ \left(-\frac{49}{48} w^4 - \frac{47}{48} w^5 \right) h + w^5 - w^4$$

The absolute stability region of the method (12) is plotted and shown in figure 4.1

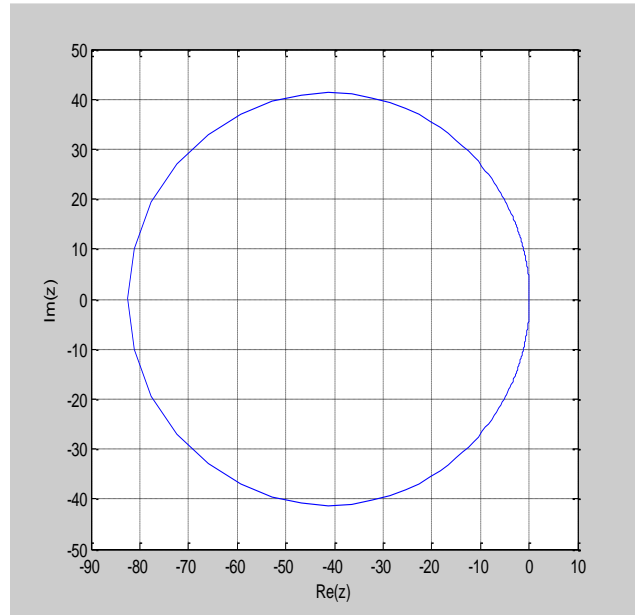


Figure 4.1 Region of Absolute Stability of the method (12).

Problem 1

The SIR model is an epidemiological model that computes the theoretical number of people infected with a contagious illness in a closed population over time. The name of this class of models derived from the fact that they involve coupled equations relating the number of susceptible people $S(t)$, number of people infected $I(t)$ and the number of people who have recovered $R(t)$. This is a good and simple model for many infectious diseases including measles, mumps and rubella. The SIR model is described by the three coupled equations

$$\frac{dS}{dt} = \mu(1-S) - \beta IS$$

$$\frac{dI}{dt} = \mu I - \gamma I + \beta IS$$

$$\frac{dR}{dt} = -\mu R + \gamma I$$

where μ, γ and β are positive parameters. Defined y to be

$$y = S + I + R$$

Adding these equations gives

$$y' = \mu(1-y)$$

Taking $\mu = 0.5$ and attaching an initial condition $y(0) = 0.5$ (for a particular closed population), we obtain

$$y'(t) = 0.5(1 - y), \quad y(0) = 0.5$$

Whose analytic solution is

$$y(t) = 1 - 0.5e^{-0.5t}$$

Source: M. A Rufai, M. K. Duromola, & M. K. Ganiyu, 2016.

Problem 2

Highly stiff problem

$$y' + 4y = 20, \quad y(0) = -4, \quad y(1) = 2, \quad 0 \leq x \leq 1, \quad h = \frac{1}{100}$$

With the exact solution

$$y(x) = 5 - 3e^{-4x}$$

Source: Badmus, A. M., Yahaya, Y. A & subair, A. O. 2014.

Table 4.1 Comparing the absolute errors for problem 1

X	Error in New Method	Error in (M. A Rufai, M. K. Duromola, & M. K. Ganiyu, 2016)
0.1	1.050426e-14	1.714000e-14
0.2	2.357104e-14	3.260000e-14
0.3	3.192613e-14	4.653000e-14
0.4	4.265593e-14	5.902000e-14
0.5	4.917574e-14	7.018000e-14
0.6	5.789502e-14	8.011000e-14
0.7	6.285319e-14	8.891000e-14
0.8	6.984743e-14	9.665000e-14
0.9	7.348215e-14	1.034200e-13
1.0	7.900071e-14	1.093100e-13

Table 4.2 Comparing the absolute errors for problem 2

X	Error in new method	Error in (Badmus, A. M., Yahaya, Y. A & subair, A. O. 2014)
0.1	1.346010e-14	2.0000(-14)
0.2	3.02037e-14	3.0000(-14)
0.3	4.144460e-14	8.0000(-14)
0.4	5.576320e-14	1.2000(-12)
0.5	6.504630e-14	1.0000(-12)
0.6	7.72137e-14	2.6970(-11)
0.7	8.47740e-14	5.5800(-12)
0.8	9.50363e-14	6.2140(-10)
0.9	1.010837e-13	2.7048(-10)
1.0	1.096619e-13	1.4571(-8)

5. Conclusion

The inclusion of off grid points allowed the adoption of linear multistep procedure which circumvents the 'zero-stability barrier, up graded the order of accuracy of the methods and to obtain very low error constants. The order, error constant, consistency, zero stability and region of absolute

stability of the methods including its hybrids form was determined.

Results from numerical solutions of stiff problems, epidemiology model real life problems, decaying problems and highly stiff problems, show that these methods are superior and performed better than the methods with which we compared our result with.

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