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### A $(2n + 9)^{th}$ Degree Polynomial Approximation Method for Solving Eight Order Linear Boundary Value Problems

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#### ABSTRACT

Galerkin method, Collocation method and least squared method have been used for solving boundary value problems (BVPs) and from practical point of view almost any success of above mention method depends on the selection of basis functions. Also, Adomian decomposition and variational iteration methods have been applied for the solution of BVPs and required respectively calculating the Adomian's polynomials and Lagrange multipliers. In this paper a simple method of  $(2n + 9)^{th}$  Degree polynomial approximation solution were  $n$  is number of successive derivatives of the governing equation is suggested as an alternative of the above mentioned methods. The method consists of first obtaining from the governing equation its  $n$  successive derivatives and boundary conditions a linear differential system of  $(2n + 10)^{th}$  equations evaluated at the boundary points. Next, an approximate solution in the form of polynomial of degree  $(2n + 9)^{th}$  with  $(2n + 10)^{th}$  unknown coefficients is assumed. To determine the unknown coefficients, the assumed solution is incorporate into the linear differential system which turns to be a linear system of  $(2n + 10)^{th}$  equation with  $(2n + 10)^{th}$  unknowns which is solvable uniquely. The method is tested for  $n=5$  to four examples. It clear that from tables 3.1 to 3.4 and figures 3.1 to 3.4 the numerical outcomes agree with the exact solution and also better than some existing results in the literatures. it should be also noted that the accuracy of the method can be increased by increased  $n$ . All Numerical computation were performed using maple 2016 software.

**Keywords:** linear boundary value problems, eighth order, successive differentiation, system of linear algebraic equations, and polynomial approximation solution.

#### 1.0 INTRODUCTION

Boundary value problems (BVPs) arise in several fields of science and engineering, where the behavior of a system or process needs to be determined at specific boundary conditions. Solving higher-order BVPs can be particularly challenging due to their complex nature, often requiring advanced numerical methods for accurate and efficient solutions.

The eighth-order BVPs involve ordinary differential equations (ODEs) of high order, and they are encountered in diverse applications such as: astrophysics, hydrodynamic and hydromagnetic stability, fluid dynamics, astronomy, and electromagnetism (El-gamel & Abdrabou, 2019). The accurate and efficient solution of these problems is crucial, due that several techniques have been developed, such as: (Akram & Rehman, 2013) used reproducing kernel space method for

the solution of such problem. Mestrovic (2007) modified Adomian decomposition method was presented for the approximate solution of 8<sup>th</sup> BVPs. In (Porshokouhi *et al.*, 2011; Al-mamun, *el al.*, 2019) the authors applied Variational Iteration Method for solving eighth order boundary value problems. Further, Researched on a combination of variational iteration method and decomposition method was used for the numerical solution of such problems (Noor & Mohyud-Din, 2007). (Golbabai & Javidi, 2007) solved such problems by Homotopy perturbation method. While (Islam & Hossain, 2015) presented Galerkin method with Bernstein and Legendre polynomials as basis functions for the numerical solution of eighth order boundary value problems. Pandey, (2018) developed finite difference method for numerical solution of 8<sup>th</sup> order boundary value problems (Ogunrinde & Ojo, 2018) employed differential transformation method to solved 8<sup>th</sup>-order BVPs.

Reddy *et al.*, (2017) presented Haar wavelet collocation method (HWCM) which is robust and accurate for numerical solution of 8th order BVPs. (Hussain & Arshad, 2018) applied one dimensional differential transformation method for solving such problem

Recently, (El-gamel & Abdrabou, 2019) developed approximate solution to such problem by using sinc-Galerkin method. Kalid *et al.*, (2019) proposed Cubic B-splines method for the solution of eighth order boundary value problem. Johnston *et al.*, (2020) proposed least-squares method for solution of eight orders by using the theory of functional connections. In Amin *et al.*, (2021) the Authors applied Haar technique to solved both nonlinear and linear eight-order BVPs. (Haq & Sohaib, 2021) used Legendre wavelet collocation method to solved high-order BVPs numerically. Pandey, (2022) used a novel finite difference

method, derived from the splitting coupled equations approach for the numerical solution of seventh order BVPs. In Raji *et al.*, (2023) the authors applied First-kind Chebyshev polynomials as a basis functions for the approximate solution of eighth order BVPs. Mentioned from the above papers, in this research, we aim to investigate the application and effectiveness of the  $(2n^{th}+9)$ -Degree Polynomial Approach in solving eighth-order linear boundary value problems. We will compare the results obtained through this approach with those obtained from other developed numerical methods by some previous authors to highlight its advantages and accuracy.

However, in this paper, the brief description of the method is present in Section 2. In Section 3, numerical examples for linear BVPs are taken into account to validate the suggested approach while Section 4 summarizes the paper's conclusion.

## 2.0 METHODOLOGY

In this paper, linear eighth-order boundary value problems are consider in the form of;

$$y^{(8)}(x) = f(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)}, y^{(6)}, y^{(7)}) \tag{2.1}$$

Subject to the following boundary conditions:

$$\left. \begin{aligned} y(0) &= \alpha_0 & y(b) &= \beta_0 \\ y^{(1)}(0) &= \alpha_1 & y^{(1)}(b) &= \beta_1 \\ y^{(2)}(0) &= \alpha_2 & y^{(2)}(b) &= \beta_2 \\ y^{(3)}(0) &= \alpha_3 & y^{(3)}(b) &= \beta_3 \end{aligned} \right\} \tag{2.2}$$

where  $f$  is continuous functions on the interval  $[0, b]$ ,  $\alpha_i, \beta_i, i=0, 1,2,3$  are finite real numbers.

The successive differentiation of eq. (2.1) n times are obtained as follows:

$$y^{(8+1)}(x) = f'(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)}, y^{(6)}, y^{(7)}) = f_1(x, y, y', y'', y''', y^{iv}, y^v, \dots, y^{(8)}) \tag{2.3}$$

$$y^{(8+2)}(x) = f''(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)}, y^{(6)}, y^{(7)}) = f_2(x, y, y', y'', y''', y^{iv}, y^v, \dots, y^{(9)}) \tag{2.4}$$

$$y^{(8+3)}(x) = f'''(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)}, y^{(6)}, y^{(7)}) = f_3(x, y, y', y'', y''', y^{iv}, y^v, \dots, y^{(10)}) \tag{2.5}$$

⋮  
⋮  
⋮

$$y^{(8+n)}(x) = f^n(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)}, y^{(6)}, y^{(7)}) = f_n(x, y, y', y'', y''', y^{iv}, y^v, \dots, y^{(8+(n-1))}) \tag{2.6}$$

After evaluating each of the n linear equation state above at 0 and b, the linear differential system of equations was obtained as follows:

$$\begin{aligned} y^{(8+1)}(0) &= f_1(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(8)}) |_{x=0} \\ y^{(8+1)}(b) &= f_1(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(8)}) |_{x=b} \\ y^{(8+2)}(0) &= f_2(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(9)}) |_{x=0} \\ y^{(8+2)}(b) &= f_2(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(9)}) |_{x=b} \\ y^{(8+3)}(0) &= f_3(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(10)}) |_{x=0} \end{aligned}$$

$$\begin{aligned}
 y^{(8+3)}(b) &= f_3(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(10)})|_{x=b} \\
 &\vdots \\
 &\vdots \\
 &\vdots \\
 y^{(8+n)}(0) &= f_n(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(8+(n-1))})|_{x=0} \\
 y^{(8+n)}(b) &= f_n(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(8+(n-1))})|_{x=b}
 \end{aligned}$$

Hence, n successive derivatives of the governing equation evaluated at 0 and b give a total of 2n<sup>th</sup> linear differential equation evaluated at 0 and b.

Combining the boundary conditions, the governing equation evaluated at 0 and b together with the above obtained 2n equation, we have the following linear differential systems of (2n<sup>th</sup>+10) equations evaluated at 0 and b as follows:

$$\left. \begin{aligned}
 &y(0) = \alpha_0 \\
 &y(b) = \beta_0 \\
 &y^{(1)}(0) = \alpha_1 \\
 &y^{(1)}(b) = \beta_1 \\
 &y^{(2)}(0) = \alpha_2 \\
 &y^{(2)}(b) = \beta_2 \\
 &y^{(3)}(0) = \alpha_3 \\
 &y^{(3)}(b) = \beta_3 \\
 &y^{(8)}(0) = f_0(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(7)})|_{x=0} \\
 &y^{(8)}(b) = f_0(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(7)})|_{x=b} \\
 &y^{(8+1)}(0) = f_1(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(8)})|_{x=0} \\
 &y^{(8+1)}(b) = f_1(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(8)})|_{x=b} \\
 &y^{(8+2)}(0) = f_2(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(9)})|_{x=0} \\
 &y^{(8+2)}(b) = f_2(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(9)})|_{x=b} \\
 &y^{(8+3)}(0) = f_3(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(10)})|_{x=0} \\
 &y^{(8+3)}(b) = f_3(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(10)})|_{x=b} \\
 &\vdots \\
 &\vdots \\
 &\vdots \\
 &y^{(8+n)}(0) = f_n(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(8+(n-1))})|_{x=0} \\
 &y^{(8+n)}(b) = f_n(x, y, y', y'', y''', y^{iv}, y^v, y^{vi}, \dots, y^{(8+(n-1))})|_{x=b}
 \end{aligned} \right\} \tag{2.7}$$

Hence, let assumed the approximate solution in the form:

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots + a_{2n+9}x^{2n+9} \tag{2.8}$$

To determine the unknown constants  $a_0, a_1, a_3, \dots, a_{2n+9}$  we incorporated equation (2.8) in to the systems of equation (2.7) which results into a linear system of (2n<sup>th</sup> +10) equations with (2n<sup>th</sup> +10) unknowns which can be solved uniquely.

### 3.0 NUMERICAL EXAMPLES

To determine the applicability of the suggested approach, linear BVPs are considered.

**Example 3.1.:** Consider the Linear BVP, (Khalid *et al.*, 2019; Amin *et al.*, 2021; Raji *et al.*, 2023)

$$y^{(8)}(x) + xy(x) = -e^x(48 + 15x + x^3) \quad 0 \leq x \leq 1 \quad (3.1)$$

Subject to the following boundary conditions  
 $y(0) = 0, y^{(1)}(0) = 1, y^{(2)}(0) = 0, y^{(3)}(0) = -3$

$$y(1) = 0, y^{(1)}(1) = -e, y^{(2)}(1) = -4e, y^{(3)}(1) = -9e \quad (3.2)$$

With the exact solution  $y(x) = x(1 - x)e^x$

$$\left. \begin{aligned} y^{(9)}(0) &= -63 \\ y^{(9)}(1) &= -81e \\ y^{(10)}(0) &= -80 \\ y^{(10)}(1) &= -100e \\ y^{(11)}(0) &= -99 \\ y^{(11)}(1) &= -121e \\ y^{(12)}(0) &= -120 \\ y^{(12)}(1) + y^{(4)}(1) &= -160e \\ y^{(13)}(0) + 5y^{(4)}(0) &= -183 \\ y^{(13)}(1) + y^{(5)}(1) + 5y^{(4)}(1) &= -274e \\ y^{(8)}(0) &= -48 \\ y^{(8)}(1) &= -64e \\ y^{(3)}(0) &= -3 \\ y^{(3)}(1) &= -9e \\ y^{(2)}(0) &= 0 \\ y^{(2)}(1) &= -4e \\ y^{(1)}(0) &= 1 \\ y^{(1)}(1) &= -e \\ y(0) &= 0 \\ y(1) &= 0 \end{aligned} \right\}$$

$$\begin{aligned} &362880a_9 = -63 \\ 362880a_9 + 3628800a_{10} + 19958400a_{11} + 79833600a_{12} \\ &+ 259459200a_{13} + 726485760a_{14} \\ &+ 1816214400a_{15} + 4151347200a_{16} \\ &+ 8821612800a_{17} + 17643225600a_{18} \\ &+ 33522128640a_{19} = -81e \\ &3628800a_{10} = -80 \\ 3628800a_{10} + 39916800a_{11} + 239500800a_{12} \\ &+ 1037836800a_{13} + 3632428800a_{14} \\ &+ 10897286400a_{15} \\ &+ 29059430400a_{16} \\ &+ 70572902400a_{17} \\ &+ 158789030400a_{18} \\ &+ 335212286400a_{19} = -100e \\ 39916800a_{11} &= -99 \end{aligned}$$

After successive differentiation eq.(3.1) five times and the results evaluated at 0 and 1 as well as the governing equation evaluated at 0 and 1 together with the boundary conditions, the linear differential system of equations evaluates at 0 and 1 is obtained as follows:

$$(3.3)$$

Since the linear differential systems of 20 equations mentioned above was obtained. One can suppose that the approximate solution in the form:

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + a_{10}x^{10} + a_{11}x^{11} + a_{12}x^{12} + a_{13}x^{13} + a_{14}x^{14} + a_{15}x^{15} + a_{16}x^{16} + a_{17}x^{17} + a_{18}x^{18} + a_{19}x^{19} \quad (3.4)$$

To determine the unknown constants  $a_0, a_1, a_3, \dots, a_{19}$  the equation (3.4) will be incorporated in to equation. (3.3) and which results into a linear system of twenty equations with twenty unknowns  $a_0, a_1, a_3, \dots, a_{19}$  and can be solved uniquely and can written as:

$$\begin{aligned} &39916800a_{11} + 479001600a_{12} + 3113510400a_{13} \\ &+ 14529715200a_{14} \\ &+ 54486432000a_{15} \\ &+ 174356582400a_{16} \\ &+ 494010316800a_{17} \\ &+ 1270312243200a_{18} \\ &+ 3016991577600a_{19} = 121e \\ &479001600a_{12} = -120 \\ 24a_4 + 120a_5 + 360a_6 + 840a_7 + 1680a_8 + 3024a_9 \\ &+ 5040a_{10} + 7920a_{11} \\ &+ 479013480a_{12} + 6227037960a_{13} \\ &+ 43589169624a_{14} \\ &+ 217945760760a_{15} \\ &+ 871782955680a_{16} \\ &+ 2964061957920a_{17} \\ &+ 8892185775840a_{18} \\ &+ 24135932713824a_{19} = -106e \\ &6227020800a_{13} + 120a_4 = -183 \end{aligned}$$

$$\begin{aligned}
 &120a_4 + 720a_5 + 2520a_6 + 6720a_7 + 15120a_8 \\
 &\quad + 30240a_9 + 55440a_{10} + 95040a_{11} \\
 &\quad + 154440a_{12} + 6227261040a_{13} \\
 &\quad + 87178651560a_{14} \\
 &\quad + 653837708160a_{15} \\
 &\quad + 3487132390560a_{16} \\
 &\quad + 14820310532160a_{17} \\
 &\quad + 53353115609760a_{18} \\
 &\quad + 168951530206080a_{19} = -274e \\
 &40320a_8 = -48 \\
 &40320a_8 + 362880a_9 + 1814400a_{10} + 6652800a_{11} \\
 &\quad + 19958400a_{12} + 51891840a_{13} \\
 &\quad + 121080960a_{14} + 259459200a_{15} \\
 &\quad + 518918400a_{16} + 980179200a_{17} \\
 &\quad + 1764322560a_{18} + 3047466240a_{19} \\
 &= -64e \\
 &6a_3 = -3
 \end{aligned}$$

$$\begin{aligned}
 &6a_3 + 24a_4 + 60a_5 + 120a_6 + 210a_7 + 336a_8 + 504a_9 \\
 &\quad + 720a_{10} + 990a_{11} + 1320a_{12} \\
 &\quad + 1716a_{13} + 2184a_{14} + 2730a_{15} \\
 &\quad + 3360a_{16} + 4080a_{17} + 4896a_{18} \\
 &\quad + 5814a_{19} = -9e \\
 &2a_2 = 0 \\
 &2a_2 + 6a_3 + 12a_4 + 20a_5 + 30a_6 + 42a_7 + 56a_8 + 72a_9 \\
 &\quad + 90a_{10} + 110a_{11} + 132a_{12} + 156a_{13} \\
 &\quad + 182a_{14} + 210a_{15} + 240a_{16} + 272a_{17} \\
 &\quad + 306a_{18} + 342a_{19} = -4e \\
 &a_1 = 1 \\
 &a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 + 7a_7 + 8a_8 + 9a_9 \\
 &\quad + 10a_{10} + 11a_{11} + 12a_{12} + 13a_{13} \\
 &\quad + 14a_{14} + 15a_{15} + 16a_{16} + 17a_{17} \\
 &\quad + 18a_{18} + 19a_{19} = -e \\
 &a_0 = 0 \\
 &a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} \\
 &\quad + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + a_{16} \\
 &\quad + a_{17} + a_{18} + a_{19} = 0
 \end{aligned} \tag{3.5}$$

After solving equation (3.5), the following unknown constants were obtained:

$$\begin{aligned}
 &a_0 = 0, a_1 = 1, a_2 = 0, a_3 = -0.5000000001, a_4 \\
 &= -0.3333333143, a_5 \\
 &= -0.1250000560, a_6 \\
 &= -0.03333327943, a_7 \\
 &= -0.00694446138, a_8 \\
 &= -0.001190476190, a_9 \\
 &= -0.0001736111111, a_{10} \\
 &= -0.00002204585538, a_{11} \\
 &= -0.000002480158731, a_{12} \\
 &= -2.505210839 \times 10^{-7}, a_{13} \\
 &= -2.296443306 \times 10^{-8}, a_{14} \\
 &= -1.927061095 \times 10^{-9}, a_{15} \\
 &= -1.4917711 \times 10^{-10}, a_{16} \\
 &= -1.064447919 \times 10^{-11}, a_{17} \\
 &= -7.5400590 \times 10^{-13}, a_{18} \\
 &= -3.20268314 \times 10^{-14}, a_{19} \\
 &= -5.04373306 \times 10^{-15}
 \end{aligned}$$

Therefore, the approximate solution is given as;

$$\begin{aligned}
 y(x) = &x - 0.5000000001x^3 - 0.3333333143x^4 - 0.1250000560x^5 - 0.03333327943x^6 \\
 &- 0.00694446138x^7 - 0.001190476190x^8 - 0.0001736111111x^9 \\
 &- 0.00002204585538x^{10} - 0.000002480158731x^{11} - 2.505210839 \times 10^{-7}x^{12} \\
 &- 2.296443306 \times 10^{-8}x^{13} - 1.927061095 \times 10^{-9}x^{14} - 1.491771100 \times 10^{-10}x^{15} \\
 &- 1.064447919 \times 10^{-11}x^{16} - 7.540059000 \times 10^{-13}x^{17} - 3.202683140 \times 10^{-14}x^{18} \\
 &- 5.043733060 \times 10^{-15}x^{19}
 \end{aligned}$$

**Table 3.1: Comparison of Numerical Solutions for Example 3.1**

x	Analytical Solution	Approximate Solution	Absolute Error		
			(Present Solution)	Khalid <i>et al.</i> , (2019)	Raji <i>et al.</i> , (2023)

0	0	0	0	0	5.022E-7
0.1	0.09946538262	0.09946538264	2.0E-11	4.13E-5	1.025E-6
0.2	0.1954244413	0.1954244414	1.0E-10	1.87E-4	1.467E-6
0.3	0.2834703497	0.2834703497	0	4.43E-4	1.464E-6
0.4	0.3580379275	0.3580379276	1.0E-10	7.48E-4	9.146E-7
0.5	0.4121803178	0.4121803179	1.0E-10	9.87E-4	6.128E-9
0.6	0.4373085123	0.4373085120	3.0E-10	1.04E-4	8.994E-7
0.7	0.4228880685	0.4228880687	2.0E-10	8.68E-4	1.442E-6
0.8	0.3560865485	0.3560865485	1.0E-10	5.15E-4	1.437E-6
0.9	0.2213642800	0.2213642799	1.0E-10	1.54 E-4	9.881E-7
1	0	9.840172065E-11	9.840172065E-11	0	4.563E-7

Table 3.1 shows a comparison between the exact solution and approximate solution, it also shows the absolute errors obtained by using the suggested approach and by of (Khalid *et al.*, 2019; Raji *et al.*, 2023).

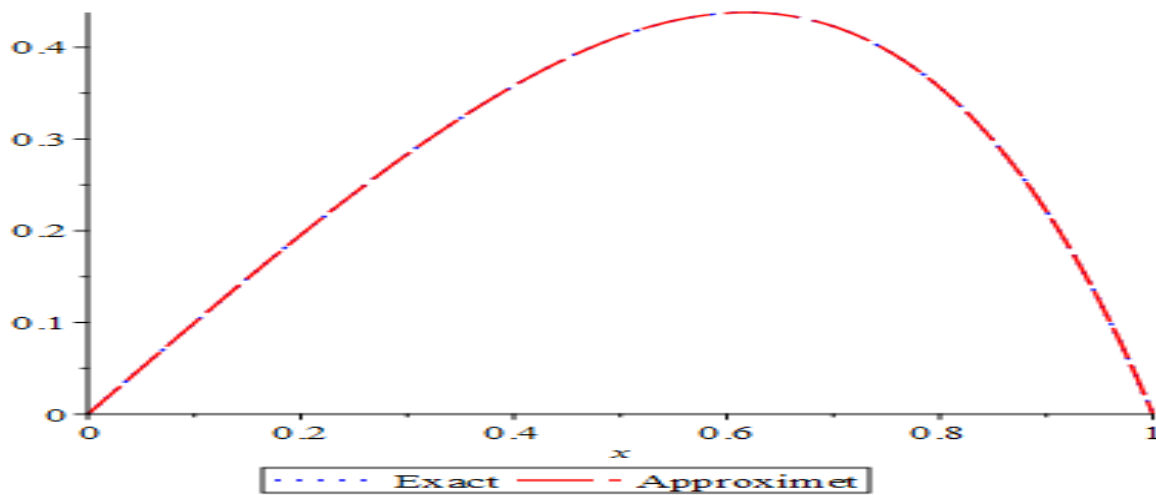


Figure 3.1: Graphical Comparison Between Exact Solution and Numerical Solution for Example 3.1

**Example 3.2:** Consider the Linear BVP (Reddy *et al.*, 2017; Khalid *et al.*, 2019).

$$y^{(8)}(x) - y(x) = -8e^x \quad 0 \leq x \leq 1 \quad (3.6)$$

Subject to following the boundary conditions:

$$y(0) = 1, y^{(1)}(0) = 0, y^{(2)}(0) = -1, y^{(3)}(0) = -2$$

$$y(1) = 0, y^{(1)}(1) = -e, y^{(2)}(1) = -2e, y^{(3)}(1) = -3e \quad (3.7)$$

With the exact solution  $y(x) = (1 - x)e^x$  Following from example (3.1) the linear differential system of equations was obtained as:

$$\left. \begin{aligned} y^{(9)}(0) &= -8 \\ y^{(9)}(1) &= -9e \\ y^{(10)}(0) &= -9 \\ y^{(10)}(1) &= -10e \\ y^{(11)}(0) &= -10 \\ y^{(11)}(1) &= -11e \\ y^{(12)}(0) - y^{(4)}(0) &= -8 \\ y^{(12)}(1) - y^{(4)}(1) &= -8e \\ y^{(13)}(0) - y^{(5)}(0) &= -8 \\ y^{(13)}(1) - y^{(5)}(1) &= -8e \\ y^{(8)}(0) &= -7 \\ y^{(8)}(1) &= -8e \\ y^{(3)}(0) &= -2 \\ y^{(3)}(1) &= -3e \\ y^{(2)}(0) &= -1 \\ y^{(2)}(1) &= -2e \\ y^{(1)}(0) &= 0 \\ y^{(1)}(1) &= -e \\ y(0) &= 1 \\ y(1) &= 0 \end{aligned} \right\}$$

$$\begin{aligned} 362880a_9 &= -8 \\ 362880a_9 + 3628800a_{10} + 19958400a_{11} \\ &+ 79833600a_{12} + 259459200a_{13} \\ &+ 726485760a_{14} + 1816214400a_{15} \\ &+ 4151347200a_{16} \\ &+ 8821612800a_{17} \\ &+ 17643225600a_{18} \\ &+ 33522128640a_{19} = -9e \\ 3628800a_{10} &= -9 \\ 3628800a_{10} + 39916800a_{11} + 239500800a_{12} \\ &+ 1037836800a_{13} \\ &+ 3632428800a_{14} \\ &+ 10897286400a_{15} \\ &+ 29059430400a_{16} \\ &+ 70572902400a_{17} \\ &+ 158789030400a_{18} \\ &+ 335212286400a_{19} = -10e \\ 39916800a_{11} &= -10 \\ 39916800a_{11} + 479001600a_{12} + 3113510400a_{13} \\ &+ 14529715200a_{14} \\ &+ 54486432000a_{15} \\ &+ 174356582400a_{16} \\ &+ 494010316800a_{17} \\ &+ 1270312243200a_{18} \\ &+ 3016991577600a_{19} = 11e \\ 479001600a_{12} - 24a_4 &= -8 \end{aligned}$$

(3.8)

Since the linear differential systems of 20 equations mentioned above was obtained. One can suppose that the approximate solution  $y(x)$  can be writing in the form:

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + a_{10}x^{10} + a_{11}x^{11} + a_{12}x^{12} + a_{13}x^{13} + a_{14}x^{14} + a_{15}x^{15} + a_{16}x^{16} + a_{17}x^{17} + a_{18}x^{18} + a_{19}x^{19} \tag{3.9}$$

To determine the unknown constants  $a_0, a_1, a_3, \dots, a_{19}$  the equation (3.9) will be incorporated in to equation (3.8) and which results into a linear system of twenty equations with twenty unknowns  $a_0, a_1, a_3, \dots, a_{19}$  and can be solved uniquely and can written as:

$$\begin{aligned} -24a_4 - 120a_5 - 360a_6 - 840a_7 - 1680a_8 - 3024a_9 \\ - 5040a_{10} - 7920a_{11} \\ + 478989720a_{12} + 6227003640a_{13} \\ + 43589121576a_{14} \\ + 217945695240a_{15} \\ + 871782868320a_{16} \\ + 2964061843680a_{17} \\ + 8892185628960a_{18} \\ + 24135932527776a_{19} = -8e \\ 6227020800a_{13} - 120a_5 = -8 \\ -120a_5 - 720a_6 - 2520a_7 - 6720a_8 - 15120a_9 \\ - 30240a_{10} - 55440a_{11} - 95040a_{12} \\ + 6226866360a_{13} \\ + 87178050960a_{14} \\ + 653836823640a_{15} \\ + 3487131123840a_{16} \\ + 14820308761440a_{17} \\ + 53353113186240a_{18} \\ + 168951526950240a_{19} = -8e \\ 40320a_8 = -7 \\ 40320a_8 + 362880a_9 + 1814400a_{10} + 6652800a_{11} \\ + 19958400a_{12} + 51891840a_{13} \\ + 121080960a_{14} + 259459200a_{15} \\ + 518918400a_{16} + 980179200a_{17} \\ + 1764322560a_{18} \\ + 3047466240a_{19} = -8e \\ 6a_3 = -2 \end{aligned}$$

$$\begin{aligned}
 &6a_3 + 24a_4 + 60a_5 + 120a_6 + 210a_7 + 336a_8 + 504a_9 \\
 &\quad + 720a_{10} + 990a_{11} + 1320a_{12} \\
 &\quad + 1716a_{13} + 2184a_{14} + 2730a_{15} \\
 &\quad + 3360a_{16} + 4080a_{17} + 4896a_{18} \\
 &\quad + 5814a_{19} = -3e \\
 &\quad 2a_2 = -1 \\
 &2a_2 + 6a_3 + 12a_4 + 20a_5 + 30a_6 + 42a_7 + 56a_8 \\
 &\quad + 72a_9 + 90a_{10} + 110a_{11} + 132a_{12} \\
 &\quad + 156a_{13} + 182a_{14} + 210a_{15} \\
 &\quad + 240a_{16} + 272a_{17} + 306a_{18} \\
 &\quad + 342a_{19} = -2e
 \end{aligned}$$

After solving equation (3.10), the following unknown constants were obtained:

$$\begin{aligned}
 &a_1 = 0 \\
 &a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 + 7a_7 + 8a_8 + 9a_9 \\
 &\quad + 10a_{10} + 11a_{11} + 12a_{12} + 13a_{13} \\
 &\quad + 14a_{14} + 15a_{15} + 16a_{16} + 17a_{17} \\
 &\quad + 18a_{18} + 19a_{19} = -e \\
 &\quad a_0 = 0 \\
 &a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} \\
 &\quad + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + a_{16} \\
 &\quad + a_{17} + a_{18} + a_{19} = 0
 \end{aligned} \tag{3.10}$$

$$\begin{aligned}
 a_0 &= 1, a_1 = 0, a_2 = -0.5000000001, a_3 \\
 &= -0.3333333334, a_4 \\
 &= -0.1250000219, a_5 \\
 &= -0.03333327459, a_6 \\
 &= -0.006944496643, a_7 \\
 &= -0.001190460646, a_8 \\
 &= -0.0001736111111, a_9 \\
 &= -0.00002204585538, a_{10} \\
 &= -0.000002480158730, a_{11} \\
 &= -2.505210839 \times 10^{-7}, a_{12} \\
 &= -2.296443378 \times 10^{-8}, a_{13} \\
 &= -1.927084129 \times 10^{-9}, a_{14} \\
 &= -1.490834385 \times 10^{-10}, a_{15} \\
 &= -1.0779535 \times 10^{-11}, a_{16} \\
 &= -6.521447330 \\
 &\quad \times 10^{-13}, a_{17} \\
 &= -7.582543 \times 10^{-14}, a_{18} \\
 &= 5.23507691 \times 10^{-15}, a_{19} \\
 &= -1.031065429 \times 10^{-15}
 \end{aligned}$$

Therefore, the approximate solution is given as

$$\begin{aligned}
 y(x) &= 1 - 0.5000000000x^2 - 0.3333333334x^3 - 0.1250000219x^4 - 0.03333327459x^5 \\
 &\quad - 0.006944496643x^6 - 0.001190460646x^7 - 0.0001736111111x^8 \\
 &\quad - 0.00002204585538x^9 - 0.000002480158730x^{10} - 2.505210839 \times 10^{-7}x^{11} \\
 &\quad - 2.296443378 \times 10^{-8}x^{12} - 1.927084129 \times 10^{-9}x^{13} - 1.490834385 \times 10^{-10}x^{14} \\
 &\quad - 1.0779535 \times 10^{-11}x^{15} - 6.521447330 \times 10^{-13}x^{16} - 7.582543 \times 10^{-14}x^{17} \\
 &\quad + 5.23507691 \times 10^{-15}x^{18} - 1.031065429 \times 10^{-15}x^{19}
 \end{aligned}$$

**Table 3.2: Comparison of Numerical Solutions for Example 3.2**

x	Analytical Solution	Approximate Solution	Absolute Error		
			Present solution	(Reddy <i>et al.</i> , 2017)	(Khalid <i>et al.</i> , 2019)
0	1	1	0	6.3E-11	0
0.1	0.9946538262	0.9946538264	2.0 E-10	6.5E-10	1.38 E-5
0.2	0.9771222064	0.9771222067	3.0 E-10	2.0E-9	5.76 E-5
0.3	0.9449011656	0.9449011652	4.0 E-10	3.3E-9	1.28 E-4
0.4	0.8950948188	0.8950948183	5.0 E-10	3.9E-9	2.06 E-4
0.5	0.8243606355	0.8243606350	5.0 E-10	3.4E-9	2.62 E-4

0.6	0.7288475200	0.7288475200	0	2.0E-9	2.71 E-4
0.7	0.6041258121	0.6041258118	3.0 E-10	6.9E-10	2.22 E-4
0.8	0.4451081856	0.4451081856	0	6.9E-10	1.30 E-4
0.9	0.2459603111	0.2459603111	1.0 E-10	7.6E-11	3.89 E-5
1	0	0	1.23 E-10		0

Table 3.2 shows a comparison between the exact solution and approximate solution, it also shows the absolute errors obtained by using the suggested approach and by (Reddy *et al.*, 2017; Khalid *et al.*, 2019)

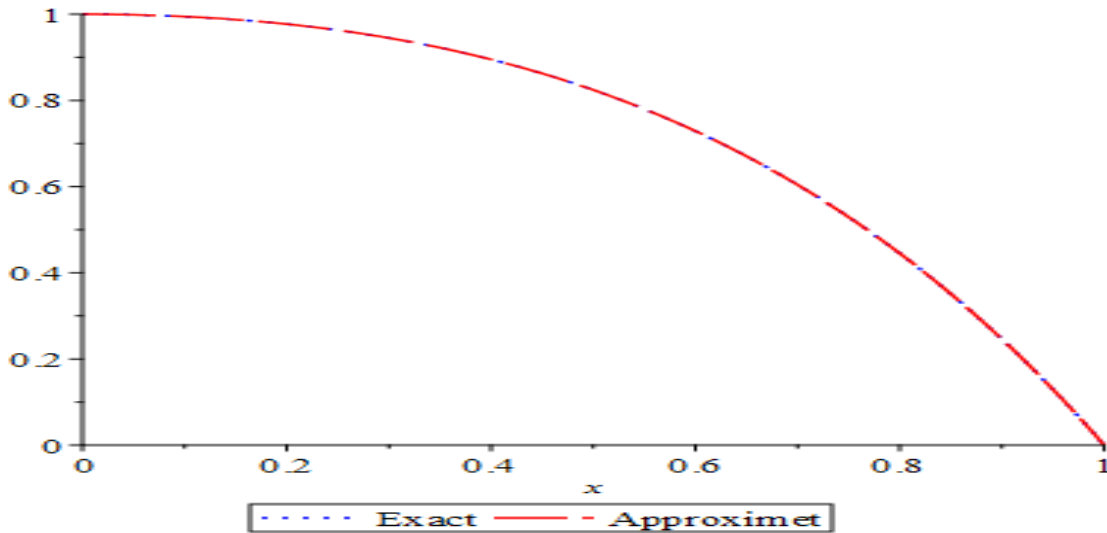


Figure 3.2: Graphical Comparison Between Exact Solution and Numerical Solution for Example 3.2

**Example3.3:** Consider the Linear BVP. ( Elahi *et al.*, 2016; Raji *et al.*, 2023)

$$y^{(8)}(x) + y^{(7)}(x) + 2y^{(6)}(x) + 2y^{(5)}(x) + 2y^{(4)}(x) + 2y^{(3)}(x) + 2y^{(2)}(x) + y^{(1)}(x) + y(x) = 14 \cos x - 16 \sin x - 4x \sin x \quad 0 \leq x \leq 1 \tag{3.11}$$

Subject to the following boundary conditions

$$\begin{aligned} y(0) = 0, y^{(1)}(0) = -1, y^{(2)}(0) = 0, y^{(3)}(0) = 7 \\ y(1) = 0, y^{(1)}(1) = 2 \sin 1, y^{(2)}(1) = 4 \cos 1 + 2 \sin 1, y^{(3)}(1) = 6 \cos 1 - 6 \sin 1 \end{aligned} \tag{3.12}$$

With the exact solution  $y(x) = (x^2 - 1) \sin x$   
Following from example (3.1) the linear differential system of equations was obtained as:

$$\begin{aligned}
 & y^{(9)}(0) + y^{(8)}(0) + 2y^{(7)}(0) + 2y^{(6)}(0) + 2y^{(5)}(0) + 2y^{(4)}(0) = -29 \\
 & y^{(9)}(1) + y^{(8)}(1) + 2y^{(7)}(1) + 2y^{(6)}(1) + 2y^{(5)}(1) + 2y^{(4)}(1) = -36 \cos 1 - 10 \sin 1 \\
 & y^{(10)}(0) + y^{(9)}(0) + 2y^{(8)}(0) + 2y^{(7)}(0) + 2y^{(6)}(0) + 2y^{(5)}(0) + 2y^{(4)}(0) = -29 \\
 & y^{(10)}(1) + y^{(9)}(1) + 2y^{(8)}(1) + 2y^{(7)}(1) + 2y^{(6)}(1) + 2y^{(5)}(1) + 2y^{(4)}(1) = -32 \cos 1 + 24 \sin 1 \\
 & y^{(11)}(0) + y^{(10)}(0) + 2y^{(9)}(0) + 2y^{(8)}(0) + 2y^{(7)}(0) + 2y^{(6)}(0) + 2y^{(5)}(0) + y^{(4)}(0) = 9 \\
 & y^{(11)}(1) + y^{(10)}(1) + 2y^{(9)}(1) + 2y^{(8)}(1) + 2y^{(7)}(1) + 2y^{(6)}(1) + 2y^{(5)}(1) + y^{(4)}(1) = 14 \cos 1 + 32 \sin 1 \\
 & y^{(12)}(0) + y^{(11)}(0) + 2y^{(10)}(0) + 2y^{(9)}(0) + 2y^{(8)}(0) + 2y^{(7)}(0) + 2y^{(6)}(0) + y^{(5)}(0) + y^{(4)}(0) = 30 \\
 & y^{(12)}(1) + y^{(11)}(1) + 2y^{(10)}(1) + 2y^{(9)}(1) + 2y^{(8)}(1) + 2y^{(7)}(1) + 2y^{(6)}(1) + y^{(5)}(1) + y^{(4)}(1) = 30 \cos 1 - 20 \sin 1 \\
 & y^{(13)}(0) + y^{(12)}(0) + 2y^{(11)}(0) + 2y^{(10)}(0) + 2y^{(9)}(0) + 2y^{(8)}(0) + 2y^{(7)}(0) + y^{(6)}(0) + y^{(5)}(0) = -16 \\
 & y^{(13)}(1) + y^{(12)}(1) + 2y^{(11)}(1) + 2y^{(10)}(1) + 2y^{(9)}(1) + 2y^{(8)}(1) + 2y^{(7)}(1) + y^{(6)}(1) + y^{(5)}(1) = -20 \cos 1 - 34 \sin 1 \\
 & y^{(8)}(0) + y^{(7)}(0) + 2y^{(6)}(0) + 2y^{(5)}(0) + 2y^{(4)}(0) = 1 \\
 & y^{(8)}(1) + y^{(7)}(1) + 2y^{(6)}(1) + 2y^{(5)}(1) + 2y^{(4)}(1) = -6 \cos 1 - 14 \sin 1 \\
 & y^{(3)}(0) = 7 \\
 & y^{(3)}(1) = 6 \cos 1 - 6 \sin 1 \\
 & y^{(2)}(0) = 0 \\
 & y^{(2)}(1) = 4 \cos 1 + 2 \sin 1 \\
 & y^{(1)}(0) = -1 \\
 & y^{(1)}(1) = 2 \sin 1 \\
 & y(0) = 0 \\
 & y(1) = 0
 \end{aligned}
 \tag{3.13}$$

Since the linear differential systems of 20 equations mentioned above was obtained. One can suppose that the approximate solution  $y(x)$  can be writing in the form:

$$\begin{aligned}
 y(x) = & a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \\
 & a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + \\
 & a_{10}x^{10} + a_{11}x^{11} + a_{12}x^{12} + a_{13}x^{13} + \\
 & a_{14}x^{14} + a_{15}x^{15} + a_{16}x^{16} + a_{17}x^{17} + \\
 & a_{18}x^{18} + a_{19}x^{19}
 \end{aligned}
 \tag{3.14}$$

To determine the unknown constants  $a_0, a_1, a_3, \dots, a_{19}$  the equation (3.14) will be incorporated in to equation (3.13) and which results into a linear system of twenty equations with twenty unknowns  $a_0, a_1, a_3, \dots, a_{19}$  and can be solved uniquely and can written as:

$$\begin{aligned}
 362880a_9 + 40320a_8 + 10080a_7 + 1440a_6 \\
 + 240a_5 + 48a_4 = -29
 \end{aligned}$$

$$\begin{aligned}
 & 37119552768a_{19} + 19757269440a_{18} \\
 & + 10017248640a_{17} \\
 & + 4798248000a_{16} \\
 & + 2148531840a_{15} \\
 & + 887014128a_{14} \\
 & + 331462560a_{13} \\
 & + 109319760a_{12} \\
 & + 30729600a_{11} \\
 & + 7025760a_{10} + 1245888a_9 \\
 & + 178080a_8 + 26880a_7 \\
 & + 3600a_6 + 480a_5 + 48a_4 \\
 & = -36 \cos 1 - 10 \sin 1 \\
 & 3628800a_{10} + 362880a_9 + 80640a_8 \\
 & + 10080a_7 + 1440a_6 + 240a_5 \\
 & + 48a_4 = -29 \\
 & 375388305408a_{19} + 180310622400a_{18} \\
 & + 81570330240a_{17} \\
 & + 34376596800a_{16} \\
 & + 13305277440a_{15} \\
 & + 4640523888a_{14} \\
 & + 1421191200a_{13} + 368778960a_{12} \\
 & + 77299200a_{11} + 12468960a_{10} \\
 & + 1608768a_9 + 218400a_8 \\
 & + 26880a_7 + 3600a_6 + 480a_5 \\
 & + 48a_4 = 24 \sin 1 - 32 \cos 1 \\
 & 39916800a_{11} + 3628800a_{10} + 725760a_9 + 80640a_8 \\
 & + 10080a_7 + 1440a_6 + 240a_5 \\
 & + 24a_4 = 9
 \end{aligned}$$

$$\begin{aligned}
 &3425901918624a_{19} + 1468266017760a_{18} \\
 &\quad + 584402202720a_{17} \\
 &\quad + 212884482720a_{16} \\
 &\quad + 69607891080a_{15} \\
 &\quad + 19896700824a_{14} \\
 &\quad + 4794143640a_{13} + 927602280a_{12} \\
 &\quad + 137166480a_{11} + 16092720a_{10} \\
 &\quad + 1968248a_9 + 216720a_8 \\
 &\quad + 26040a_7 + 3240a_6 + 360a_5 \\
 &\quad + 24a_4 = 32 \sin 1 + 14 \cos 1 \\
 &479001600a_{12} + 39916800a_{11} + 7257600a_{10} \\
 &\quad + 725760a_9 + 80640a_8 + 10080a_7 \\
 &\quad + 1440a_6 + 120a_5 + 24a_4 = 30 \\
 &27897054430464a_{19} + 10519239722400a_{18} \\
 &\quad + 361903663360a_{17} \\
 &\quad + 1113726300960a_{16} \\
 &\quad + 298450545120a_{15} \\
 &\quad + 67118034984a_{14} \\
 &\quad + 12058846800a_{13} \\
 &\quad + 1646009640a_{12} + 177027840a_{11} \\
 &\quad + 19691280a_{10} + 1953504a_9 \\
 &\quad + 210000a_8 + 23520a_7 + 2520a_6 \\
 &\quad + 240a_5 + 24a_4 \\
 &= 30 \cos 1 - 20 \sin 1 \\
 &6227020800a_{13} + 479001600a_{12} + 79833600a_{11} \\
 &\quad + 7257600a_{10} + 725760a_9 \\
 &\quad + 80640a_8 + 10080a_7 + 720a_6 \\
 &\quad + 120a_5 = -16 \\
 &199865554725600a_{19} + 65142652740480a_{18} \\
 &\quad + 18933347116320a_{17} \\
 &\quad + 4775208721920a_{16} \\
 &\quad + 1006770524760a_{15} \\
 &\quad + 168823855200a_{14} \\
 &\quad + 21398125320a_{13} \\
 &\quad + 2124334080a_{12} + 216604080a_{11} \\
 &\quad + 19535040a_{10} + 1890000a_9 \\
 &\quad + 188160a_8 + 17640a_7 + 1440a_6 \\
 &\quad + 120a_5 = -20 \cos 1 - 34 \sin 1 \\
 &40320a_8 + 5040a_7 + 1440a_6 + 240a_5 + 48a_4 = 1 \\
 &3343468608a_{19} + 1953650880a_{18} + 1097617920a_{17} \\
 &\quad + 589243200a_{16} + 299885040a_{15} \\
 &\quad + 143231088a_{14} + 63354720a_{13} \\
 &\quad + 25494480a_{12} + 9108000a_{11} \\
 &\quad + 2792160a_{10} + 701568a_9 \\
 &\quad + 137760a_8 + 21840a_7 + 3600a_6 \\
 &\quad + 480a_5 + 48a_4 \\
 &= -6 \cos 1 - 14 \sin 1 \\
 &6a_3 = 7
 \end{aligned}$$

Therefore, the approximate solution is given as

$$\begin{aligned}
 y(x) = &-x + 1.166666667x^3 - 9.859688147 \times 10^{-10}x^4 - 0.1749999989x^5 \\
 &- 3.16580239 \times 10^{-10}x^6 + 0.008531745971x^7 + 1.330000000 \times 10^{-11}x^8 \\
 &- 0.0002011684295x^9 - 1.500000000 \times 10^{-13}x^{10} + 0.000002780784022x^{11} \\
 &+ 1.320000000 \times 10^{-15}x^{12} - 2.521269882 \times 10^{-8}x^{13} - 7.48265095 \times 10^{-14}x^{14} \\
 &+ 1.615000000 \times 10^{-10}x^{15} - 1.221680157 \times 10^{-13}x^{16} - 7.1270607 \times 10^{-13}x^{17} \\
 &- 1.27275290 \times 10^{-14}x^{18} + 3.968577335 \times 10^{-15}x^{19}
 \end{aligned}$$

**Table 3.3 Comparison of Numerical Solutions for example 3.3**

x	Analytical Solution	Approximate Solution	Absolute Error
---	---------------------	----------------------	----------------

$$\begin{aligned}
 &6a_3 + 24a_4 + 60a_5 + 120a_6 + 210a_7 + 336a_8 + 504a_9 \\
 &\quad + 720a_{10} + 990a_{11} + 1320a_{12} \\
 &\quad + 1716a_{13} + 2184a_{14} + 2730a_{15} \\
 &\quad + 3360a_{16} + 4080a_{17} + 4896a_{18} \\
 &\quad + 5814a_{19} = 6 \cos 1 - 6 \sin 1 \\
 &2a_2 = 0 \\
 &2a_2 + 6a_3 + 12a_4 + 20a_5 + 30a_6 + 42a_7 + 56a_8 \\
 &\quad + 72a_9 + 90a_{10} + 110a_{11} + 132a_{12} \\
 &\quad + 156a_{13} + 182a_{14} + 210a_{15} \\
 &\quad + 240a_{16} + 272a_{17} + 306a_{18} \\
 &\quad + 342a_{19} = 4 \cos 1 + 2 \sin 1 \\
 &a_1 = -1 \\
 &a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 + 7a_7 + 8a_8 + 9a_9 \\
 &\quad + 10a_{10} + 11a_{11} + 12a_{12} + 13a_{13} \\
 &\quad + 14a_{14} + 15a_{15} + 16a_{16} + 17a_{17} \\
 &\quad + 18a_{18} + 19a_{19} = 2 \sin 1 \\
 &a_0 = 0 \\
 &a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + \\
 &a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + a_{16} + a_{17} + a_{18} + \\
 &a_{19} = 0
 \end{aligned}$$

(3.15)

After solving equation (3.15), the following unknown constants were obtained:

$$\begin{aligned}
 a_0 = 0, a_1 = -1, a_2 = 0, a_3 \\
 &= -0.3333333334, a_4 \\
 &= -9.859688147 \times 10^{-10}, a_5 \\
 &= -0.1749999989, a_6 \\
 &= -3.16580239 \times 10^{-10}, a_7 \\
 &= 0.008531745971, a_8 \\
 &= 1.330 \times 10^{-11}, a_9 \\
 &= -0.0002011684295, a_{10} \\
 &= -1.5 \times 10^{-13}, a_{11} \\
 &= 0.000002780784022, a_{12} \\
 &= 1.320000000 \times 10^{-15}, a_{13} \\
 &= -2.521269882 \times 10^{-8}, a_{14} \\
 &= -7.48265095 \times 10^{-14}, a_{15} \\
 &= 1.615000000 \times 10^{-10}, a_{16} \\
 &= -1.221680157 \\
 &\times 10^{-13}, a_{17} \\
 &= -7.1270607 \times 10^{-13}, a_{18} \\
 &= -1.27275290 \times 10^{-14}, a_{19} \\
 &= 3.968577335 \times 10^{-15}
 \end{aligned}$$

			Present Solution	(Elahi <i>et al.</i> , 2016)	(Raji <i>et al.</i> , 2023)
0	0	0	0	0	4.789E-9
0.1	-0.09883508248	-0.09883508248	0	5.03751 E-8	2.104E-8
0.2	-0.1907225576	-0.1907225576	0	5.1436 E-7	2.361E-7
0.3	-0.2689233881	-0.2689233881	0	1.55915 E-6	6.649E-7
0.4	-0.3271114075	-0.3271114075	0	2.71487 E-6	1.062E-6
0.5	-0.3595691540	-0.3595691539	1.0 E-10	3.26015 E-6	1.63E-6
0.6	-0.3613711830	-0.3613711828	2.0E-10	2.82218 E-6	9.140E-7
0.7	-0.3285510205	-0.3285510203	2. 0E-10	1.68491 E-6	4.944E-7
0.8	-0.2582481927	-0.2582481927	0	5.77885 E-7	1.555E-7
0.9	-0.1488321128	-0.1488321130	1.0 E-10	5.88442 E-8	1.872E-8
1	0	6.955604045 E-11	6.955604045 E-11		5.069E-9

Table 3.3 shows a comparison between the exact solution and approximate solution, it also shows the absolute errors obtained by using the suggested approach and absolute errors in (Elahi *et al.*, 2016; Raji *et al.*, 2023).

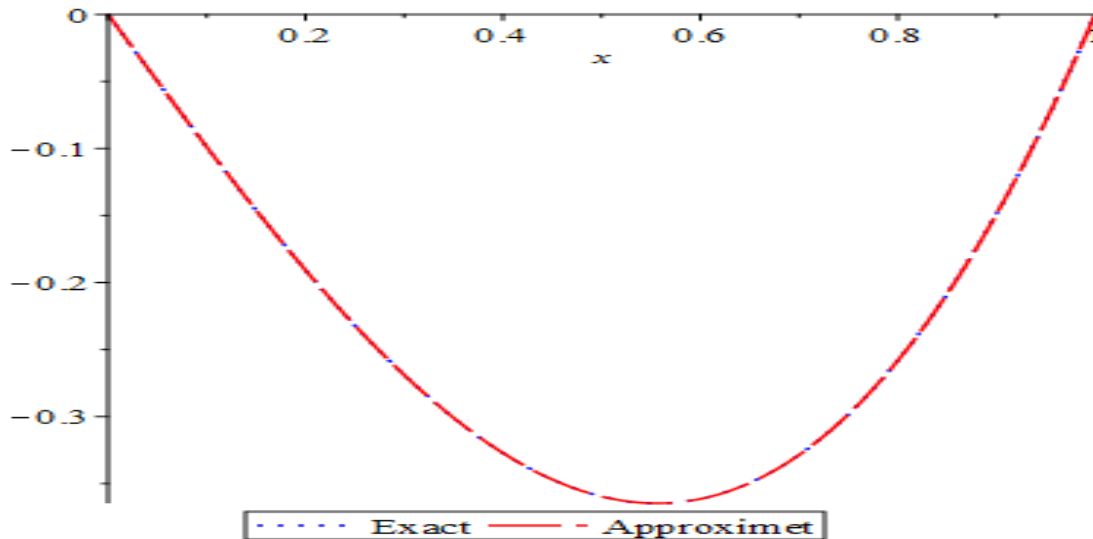


Figure 3.3: Graphical Comparison Between Exact Solution and Numerical Solution for Example 3.3

**Example3. 4:** Consider the Linear BVP. (Khalid *et al.*, 2019; Raji *et al.*, 2023)

$$y^{(8)}(x) - y(x) = -8(2x \cos x + 7 \sin x) \quad 0 \leq x \leq 1 \tag{3.16}$$

Subject to the following boundary conditions:

$$y(0) = 0, y^{(1)}(0) = -1, y^{(2)}(0) = 0, y^{(3)}(0) = 7$$

$$y(1) = 0, y^{(1)}(1) = 2 \sin 1, y^{(2)}(1) = 4 \cos 1 + 2 \sin 1, y^{(3)}(1) = 6 \cos 1 - 6 \sin 1 \tag{3.17}$$

with the exact solution  $y(x) = (x^2 - 1) \sin x$

Following from example (3.1) the linear differential system of equations was obtained as:

$$\begin{aligned}
 & y^{(9)}(0) = -71 \\
 & y^{(9)}(1) = 14 \sin 1 - 72 \cos 1 \\
 & y^{(10)}(0) = 0 \\
 & y^{(10)}(1) = 90 \sin 1 + 20 \cos 1 \\
 & y^{(11)}(0) = 111 \\
 & y^{(11)}(1) = -22 \sin 1 + 110 \cos 1 \\
 & y^{(12)}(0) - y^{(4)}(0) = 0 \\
 & y^{(12)}(1) - y^{(4)}(1) = -120 \sin 1 - 16 \cos 1 \\
 & y^{(13)}(0) - y^{(5)}(0) = -136 \\
 & y^{(13)}(1) - y^{(5)}(1) = 16 \sin 1 - 136 \cos 1 \\
 & y^{(8)}(0) = 0 \\
 & y^{(8)}(1) = -56 \sin 1 - 16 \cos 1 \\
 & y^{(3)}(0) = 7 \\
 & y^{(3)}(1) = -6 \sin 1 + 6 \cos 1 \\
 & y^{(2)}(0) = 0 \\
 & y^{(2)}(1) = 2 \sin 1 + 4 \cos 1 \\
 & y^{(1)}(0) = -1 \\
 & y^{(1)}(1) = 2 \sin 1 \\
 & y(0) = 0 \\
 & y(1) = 0
 \end{aligned}
 \tag{3.18}$$

Since the linear differential systems of 20 equations mentioned above was obtained. One can suppose that the approximate solution  $y(x)$  can be writing in the form:

$$\begin{aligned}
 y(x) = & a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \\
 & a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + \\
 & a_{10}x^{10} + a_{11}x^{11} + a_{12}x^{12} + a_{13}x^{13} + \\
 & a_{14}x^{14} + a_{15}x^{15} + a_{16}x^{16} + a_{17}x^{17} + \\
 & a_{18}x^{18} + a_{19}x^{19}
 \end{aligned}
 \tag{3.19}$$

To determine the unknown constants  $a_0, a_1, a_3, \dots, a_{19}$  the equation (3.19) will be incorporated in to equation (3.18) and which results into a linear system of twenty equations with twenty unknowns  $a_0, a_1, a_3, \dots, a_{19}$  and can be solved uniquely and can written as:

$$\begin{aligned}
 & 362880a_9 = -71 \\
 & 362880a_9 + 3628800a_{10} + 19958400a_{11} + 79833600a_{12} \\
 & + 259459200a_{13} + 726485760a_{14} \\
 & + 1816214400a_{15} + 4151347200a_{16} \\
 & + 8821612800a_{17} + 17643225600a_{18} \\
 & + 33522128640a_{19} \\
 & = -72 \cos 1 + 14 \sin 1
 \end{aligned}$$

$$\begin{aligned}
 & 3628800a_{10} = 0 \\
 & 3628800a_{10} + 39916800a_{11} + 239500800a_{12} \\
 & + 1037836800a_{13} + 3632428800a_{14} \\
 & + 10897286400a_{15} \\
 & + 29059430400a_{16} \\
 & + 70572902400a_{17} \\
 & + 158789030400a_{18} \\
 & + 335212286400a_{19} \\
 & = 90 \sin 1 + 20 \cos 1 \\
 & 39916800a_{11} = 111 \\
 & 39916800a_{11} + 479001600a_{12} + 3113510400a_{13} \\
 & + 14529715200a_{14} \\
 & + 54486432000a_{15} \\
 & + 174356582400a_{16} \\
 & + 494010316800a_{17} \\
 & + 1270312243200a_{18} \\
 & + 3016991577600a_{19} \\
 & = 110 \cos 1 - 22 \sin 1 \\
 & 479001600a_{12} - 24a_4 = 0 \\
 & -24a_4 - 120a_5 - 360a_6 - 840a_7 - 1680a_8 - 3024a_9 \\
 & - 5040a_{10} - 7920a_{11} \\
 & + 478989720a_{12} + 6227003640a_{13} \\
 & + 43589121576a_{14} \\
 & + 217945695240a_{15} \\
 & + 871782868320a_{16} \\
 & + 2964061843680a_{17} \\
 & + 8892185628960a_{18} \\
 & + 24135932527776a_{19} \\
 & = -16 \cos 1 - 120 \sin 1 \\
 & 6227020800a_{13} - 120a_5 = -136 \\
 & -120a_5 - 720a_6 - 2520a_7 - 6720a_8 - 15120a_9 \\
 & - 30240a_{10} - 55440a_{11} - 95040a_{12} \\
 & + 6226866360a_{13} + 87178050960a_{14} \\
 & + 653836823640a_{15} \\
 & + 3487131123840a_{16} \\
 & + 14820308761440a_{17} \\
 & + 53353113186240a_{18} \\
 & + 168951526950240a_{19} \\
 & = -136 \cos 1 + 16 \sin 1 \\
 & 40320a_8 = 0 \\
 & 40320a_8 + 362880a_9 + 1814400a_{10} + 6652800a_{11} \\
 & + 19958400a_{12} + 51891840a_{13} \\
 & + 121080960a_{14} + 259459200a_{15} \\
 & + 518918400a_{16} + 980179200a_{17} \\
 & + 1764322560a_{18} + 3047466240a_{19} \\
 & = -16 \cos 1 - 56 \sin 1 \\
 & 6a_3 = 7 \\
 & 6a_3 + 24a_4 + 60a_5 + 120a_6 + 210a_7 + 336a_8 + 504a_9 \\
 & + 720a_{10} + 990a_{11} + 1320a_{12} \\
 & + 1716a_{13} + 2184a_{14} + 2730a_{15} \\
 & + 3360a_{16} + 4080a_{17} + 4896a_{18} \\
 & + 5814a_{19} = 6 \cos 1 - 6 \sin 1 \\
 & 2a_2 = 0 \\
 & 2a_2 + 6a_3 + 12a_4 + 20a_5 + 30a_6 + 42a_7 + 56a_8 + 72a_9 \\
 & + 90a_{10} + 110a_{11} + 132a_{12} + 156a_{13} \\
 & + 182a_{14} + 210a_{15} + 240a_{16} + 272a_{17} \\
 & + 306a_{18} + 342a_{19} = 4 \cos 1 + 2 \sin 1 \\
 & a_1 = -1
 \end{aligned}$$

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 + 7a_7 + 8a_8 + 9a_9 + 10a_{10} + 11a_{11} + 12a_{12} + 13a_{13} + 14a_{14} + 15a_{15} + 16a_{16} + 17a_{17} + 18a_{18} + 19a_{19} = 2 \sin 1$$

$$a_0 = 0$$

After solving equation (3.21), the following unknown constants were obtained:

$$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + a_{16} + a_{17} + a_{18} + a_{19} = 0$$

(3.21)

$$a_0 = 0, a_1 = -1, a_2 = 0, a_3 = 1.166666667, a_4 = 0.00002036400257, a_5 = -0.1750774915, a_6 = 0.000105078795, a_7 = 0.008478212103, a_8 = 0, a_9 = -0.0001956569665, a_{10} = 0, a_{11} = 0.000002780784031, a_{12} = 1.020322399 \times 10^{-12}, a_{13} = -2.521419215 \times 10^{-8}, a_{14} = 0.000001675229459, a_{15} = -0.000003970732275, a_{16} = 0.000003862305277, a_{17} = -0.000001954417735, a_{18} = 5.15216267 \times 10^{-7}, a_{19} = -5.64743054 \times 10^{-8}$$

Therefore, the approximate solution is given as

$$y(x) = -x + 1.166666667x^3 + 0.00002036400257x^4 - 0.1750774915x^5 + 0.000105078795x^6 + 0.008478212103x^7 - 0.0001956569665x^9 + 0.000002780784031x^{11} - 1.020322399 \times 10^{-12}x^{12} - 2.521419215 \times 10^{-8}x^{13} + 0.000001675229459x^{14} - 0.000003970732275x^{15} + 0.000003862305277x^{16} - 0.000001954417735x^{17} + 5.15216267 \times 10^{-7}x^{18} - 5.64743054 \times 10^{-8}x^{19}$$

**Table 3.4 Comparison of Numerical Solutions for example 3.4**

x	Analytical Solution	Approximate Solution	Absolute Error		
			Present Solution	(Khalid <i>et al.</i> , 2019)	(Raji <i>et al.</i> , 2023)
0	0	0	0	0	4.196E-14
0.1	-0.09883508248	-0.09883508110	1.38 E-9	4.22 E-6	
0.2	-0.1907225576	-0.1907225438	1.38 E-8	2.71 E-5	7.716E-6
0.3	-0.2689233881	-0.2689233465	4.16 E-8	8.44 E-5	
0.4	-0.3271114075	-0.3271113356	7.19 E-8	1.70 E-4	1.324E-5
0.5	-0.3595691540	-0.3595690681	8.59 E-8	2.51 E-4	
0.6	-0.3613711830	-0.3613711095	7.35E-8	2.88 E-4	1.507E-5
0.7	-0.3285510205	-0.3285509774	4. 31E-8	2.55 E-4	
0.8	-0.2582481927	-0.2582481780	1.47 E-8	1.59 E-4	1.400E-5
0.9	-0.1488321128	-0.1488321113	1.5 E-9	5.03 E-5	
1	0	1.3004660 E-10	1.3004660 E-10	0	1.171E-5

Table 3.4 shows a comparison between the exact solution and approximate solution, it also shows the absolute errors obtained by using the suggested approach and absolute errors in (Khalid *et al.*, 2019; Raji *et al.*, 2023).

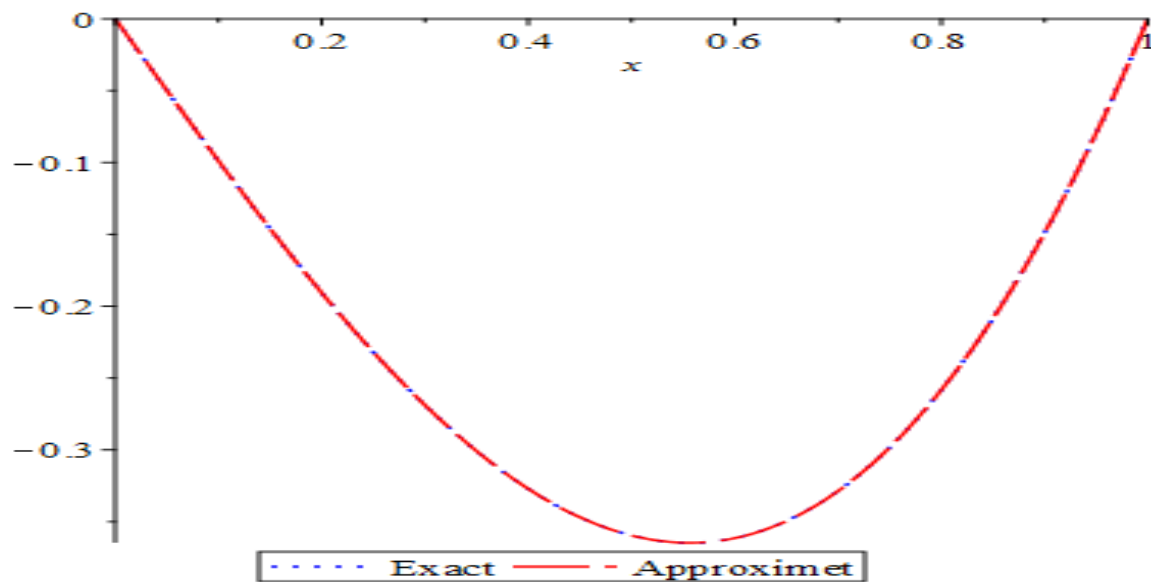


Figure 3.4: Graphical Comparison Between Exact Solution and Numerical Solution for Example 3.4

It is noted that the numerical findings shown on tables 3.1 to 3.4 are in excellent agreement with the exact solution and with some results obtained from the established techniques in the literatures. Additionally, figures 3.1 to 3.4 show that the numerical solution and with the exact solution coincide, supporting the efficacy of the suggested approach.

#### 4.0 CONCLUSION

In this work, we have used new developed  $(2n + 9)^{th}$  Degree Polynomial Approach to

obtained approximate solution of 8th-order linear BVPs. The numerical outcomes show that only five successive differentia of the governing equation lead to a high degree of accuracy. It is clear for the tables 3.1 to 3.4 that the numerical outcomes obtained by suggested approach are higher to the other exiting methods in literature. Finally, the numerical outcomes showed that the present technique is an effective mathematical tool for solving higher order BVPs.

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