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On Generators and Undirected Cayley graphs of Γ_1 -Non-Deranged Permutation Groups

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ABSTRACT

A Cayley graph $Cay(G, S)$ of a group G , where S is the set of generators of G is a graph that has a vertex for each element of G and edge $\{g, gs\}$ for $g \in G$ and $s \in S$. In this work, we study the structural properties of undirected Cayley graphs of the Γ_1 -non deranged permutation group $(G_p^{\Gamma_1})$ obtained from a modulo p function (for $p = 5$ and p is prime), and established that the generators of $G_p^{\Gamma_1}$ are not unique and also the undirected Cayley graphs $Cay(G_p^{\Gamma_1}, S)$ is not simple.

Keywords: generators; undirected Cayley graphs; subnormal series; Γ_1 -non deranged permutation group.

1.0 INTRODUCTION

The Cayley graph also known as group diagram, or color graph that encodes the abstract structure of a group was first considered for a finite group (Wang, 2017) construct several families of new self-complementary Cayley graphs of order p^4 where p is a prime and congruent to 1 modulo 8. Luchinni (2020a) enumerated the various families of Cayley graphs and digraphs. In their study, they examined the directed and undirected cases of the following three families of graphs: Cayley graphs on cyclic groups (known as circulant graphs, of square-free order), circulant graphs of arbitrary order n whose connection sets are subsets of the group of units of \mathbf{Z}_n and Cayley graphs on elementary Abelian groups. Malekmohammadi and Ashraf (2015) investigated the behavior of Cayley graphs under some graph operations, they proved that the corona, hierarchical, strong, skew, and converse skew products of Cayley graphs are again Cayley graphs under some conditions. Ejima et al. (2020a) studied the energy inverse graphs of the dihedral and symmetric groups, Aremu et al. (2017) established the Γ_1 -non deranged permutation as a group denoted by $(G_p^{\Gamma_1})$ and afterword Ibrahim et al. (2022a) studied the representation of Γ_1 -non deranged permutation group and established that character of every

$\omega_i \in G_p^{\Gamma_1}$ is not zero. Recently, Ejima et al. (2020b) investigated some algebraic properties and combinatorial structures of Subgroup graph and obtain that the subgroup graph is never bipartite. Magami et al. (2021a) applied the Γ_1 non deranged permutations in statistics by Partitioning blocks coordinate. Also, Ibrahim et al. (2022b) worked on embracing sum using ascent block of-non deranged permutation. Garba et al. (2010) established that the young tableaux of $(G_p^{\Gamma_1})$ is non standard. In this work we investigate the structural properties of the undirected Cayley graphs of Γ_1 -non deranged permutation group through the generating set of $G_p^{\Gamma_1}$. Luchinni (2020b) proved that graphs whose vertices are element of finite group is connected and classify the groups for which graphs is a planar graph.

2.0. PRELIMINARIES

2.1. Definition 1: A graph is an ordered pair $G = (V, E)$ consisting of a nonempty set V (called the vertices) and a set E (called the edges) of two-element subsets of V .

2.2. Definition 2: A directed graph G consists of a set V of vertices (or nodes) together with a set E of

ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a,b) , and the vertex b is called the terminal vertex of this edge, if there is an orientation (direction) from a to b . An undirected graph has no orientation.

2.3. Definition 3: Let G be a graph, a multigraph is a graph with multiple edges (or parallel edges).

2.4. Definition 4: Let G be a graph and v a vertex of G . The degree of v written as $\text{deg}(v)$ is the number of edges incidence with v .

2.5. Definition 5 The Cayley graph $\text{Cay}(K, S)$ of a group K with respect to set S of generators has K for its vertex set, and edge $\{g, gs\}$ where $g \in K$ and $s \in S$.

Remark 1. Cayley graph of any group have no self-loop.

Remark 2 Cayley graph of a group is not unique, since it depends on the generating set.

Lemma 1. (Handshaking Lemma) Let $G = (V, E)$ be an undirected graph with m edges. Then

$$\sum_{v \in V} \text{deg}(v) = 2m$$

2.6. Definition 6: Let Γ be non empty set of prime cardinality greater or equal to 5 such that $\Gamma \subset N$ A bijection ω on Γ of the form

$$= \left(\begin{matrix} \omega_i \\ 1 \wedge 2 3 \wedge \dots \wedge p \\ 1 \wedge (1+i)_{mp} (1+2i)_{mp} \wedge \dots \wedge (1+(p-1)i)_{mp} \end{matrix} \right)$$

is call Γ_1 -non deranged permutation. Where G_p denote the set of all Γ_1 -non deranged permutations.

Proposition 1: Aminu et al. (2022). Let $G_p^{\Gamma_1}$ be a non empty set of Γ_1 -non deranged permutations. Let \circ be the binary operation (natural composition of permutation) on $G_p^{\Gamma_1}$. If $\forall \omega \in G_p^{\Gamma_1}$ and $\omega \circ \omega^{-1}$. Then $G_p^{\Gamma_1}$ is a subgroup of S_p .

Corollary: 1 Aremu et al. (2020a). The pair $(G_p^{\Gamma_1}, \circ)$ is a group.

Remark 3: Aremu et al. (2020b). Every subgroup of $G_p^{\Gamma_1}$ is normal even if $G_p^{\Gamma_1}$ is not normal in S_p

2.7. Definition 7: Let G be a group. A subset H of G is said to be subgroup of G if the group operation of G is restricted to H .

2.8. Definition 8: A group is said to be cyclic if there exist an element $g \in G$ such that $G = \{g^n : n \in N\}$; i.e if there exist an element $g \in G$ that can generate all elements of G .

Theorem 1: Every subgroup of a cyclic group is cyclic.

2.9. Definition 9: Let G be a group. A subnormal series of G is a chain of subgroup

$$G = G_0 > G_1 > \dots > G_n$$

such that $G_i \triangleright G_{i+1}$ for all i

2.2.1 Definition 10: Let G be a group. A sequence of subgroup

$$\{I\} = G_n \triangleleft \dots \triangleleft G_2 \triangleleft G_1 \triangleleft G_0 = G$$

where each subgroup is normal in the next one as indicated is called normal series for the group.

Remark 4: Suppose $e = \{I\}$ is an identity of any group, then $\{I\} \subseteq G$ is a normal series.

Example 1. Let $G_p^{\Gamma_1}$ be a group and $G_p^{\Gamma_1} < S_p$

- i. $\{I\} \subseteq H \subseteq G_p^{\Gamma_1}$ is a normal subgroup in $G_p^{\Gamma_1}$ if $H = \{\omega_1, \omega_{p-1}\}$.
- ii. $\{I\} \subseteq G_p^{\Gamma_1} \subseteq S_p$ is a subnormal series in S_p .

By remark 3 $H = \{\omega_1, \omega_{p-1}\}$ is normal, hence $\{I\} \subseteq H \subseteq G_p^{\Gamma_1}$ is a normal series.

Since $G_p^{\Gamma_1}$ is Abelian, it implies that the left coset coincides with the right coset, so $G_p^{\Gamma_1}$ is normal, and $G_p^{\Gamma_1}$ is a subgroup of S_p . It follows that $\{I\} \subseteq G_p^{\Gamma_1} \subseteq S_p$ is a subnormal series.

Example 2: Suppose S_p is a symmetric group of prime order. Then the following is a subnormal

series in S_p $\{I\} \subseteq H \subseteq G_p^{\Gamma_1} \subseteq S_p$ where $H = \{\omega_1, \omega_{p-1}\}$.

3.0. Generators of Γ_1 -non deranged permutation group

Proposition 2. The Γ_1 -non deranged permutation group is not a cyclic subgroup of the symmetric group S_p .

proof. Γ_1 -non deranged permutation group has more than one generator, hence is not a cyclic subgroup of the symmetric group S_p . ■

The following are tables of self composition of Γ_1 -non deranged permutations for $p = 5$

| | | | |
|------------------|------------|------------|------------|
| $G_5^{\Gamma_1}$ | 2 | 3 | 4 |
| ω_1 | ω_1 | ω_1 | ω_1 |
| ω_2 | ω_4 | ω_3 | ω_1 |
| ω_3 | ω_4 | ω_2 | ω_1 |
| ω_4 | ω_1 | ω_4 | ω_1 |

From the table above $(\omega_2)^1 = \omega_2, (\omega_2)^2 = \omega_4, (\omega_2)^3 = \omega_3, (\omega_2)^4 = \omega_1$ and $(\omega_3)^1 = \omega_3, (\omega_3)^2 = \omega_4, (\omega_3)^3 = \omega_2, (\omega_3)^4 = \omega_1$. Hence both ω_2 and ω_3 are generators of $G_5^{\Gamma_1}$ So we have the generating set $S = \{\omega_2, \omega_3\}$

for $p = 7$

| | | | | | |
|------------------|------------|------------|------------|------------|------------|
| $G_7^{\Gamma_1}$ | 2 | 3 | 4 | 5 | 6 |
| ω_1 | ω_1 | ω_1 | ω_1 | ω_1 | ω_1 |
| ω_2 | ω_4 | ω_1 | ω_2 | ω_4 | ω_1 |
| ω_3 | ω_2 | ω_6 | ω_4 | ω_5 | ω_1 |
| ω_4 | ω_2 | ω_1 | ω_4 | ω_2 | ω_1 |
| ω_5 | ω_4 | ω_6 | ω_2 | ω_3 | ω_1 |
| ω_6 | ω_1 | ω_6 | ω_1 | ω_6 | ω_1 |

As seen from the above table $(\omega_3)^1 = \omega_3, (\omega_3)^2 = \omega_2, (\omega_3)^3 = \omega_6, (\omega_3)^4 = \omega_4, (\omega_3)^5 = \omega_5, (\omega_3)^6 = \omega_1$ and $(\omega_5)^1 = \omega_5, (\omega_5)^2 = \omega_4, (\omega_5)^3 = \omega_6, (\omega_5)^4 = \omega_2, (\omega_5)^5 = \omega_3, (\omega_5)^6 = \omega_1$ Therefore the generating set is $S = \{\omega_3, \omega_5\}$.

$p = 11$

| | | | | | | | | | |
|---------------------|------------|------------|------------|---------------|------------|------------|------------|------------|------------|
| $G_{11}^{\Gamma_1}$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| ω_1 | ω_1 | ω_1 | ω_1 | ω_1 | ω_1 | ω_1 | ω_1 | ω_1 | ω_1 |
| ω_2 | ω_4 | ω_8 | ω_5 | ω_{10} | ω_9 | ω_7 | ω_3 | ω_6 | ω_1 |

| | | | | | | | | | |
|---------------|------------|---------------|------------|---------------|------------|---------------|------------|---------------|------------|
| ω_3 | ω_9 | ω_5 | ω_4 | ω_1 | ω_3 | ω_9 | ω_5 | ω_4 | ω_1 |
| ω_4 | ω_5 | ω_9 | ω_3 | ω_1 | ω_4 | ω_5 | ω_9 | ω_3 | ω_1 |
| ω_5 | ω_3 | ω_4 | ω_9 | ω_1 | ω_5 | ω_3 | ω_4 | ω_9 | ω_1 |
| ω_6 | ω_3 | ω_7 | ω_9 | ω_{10} | ω_5 | ω_8 | ω_4 | ω_2 | ω_1 |
| ω_7 | ω_5 | ω_2 | ω_3 | ω_{10} | ω_4 | ω_6 | ω_9 | ω_8 | ω_1 |
| ω_8 | ω_9 | ω_6 | ω_4 | ω_{10} | ω_3 | ω_2 | ω_5 | ω_7 | ω_1 |
| ω_9 | ω_4 | ω_3 | ω_5 | ω_1 | ω_9 | ω_4 | ω_3 | ω_5 | ω_1 |
| ω_{10} | ω_1 | ω_{10} | ω_1 | ω_{10} | ω_1 | ω_{10} | ω_1 | ω_{10} | ω_1 |

As seen from the table above, for $G_{11}^{\Gamma_1}$ the generating set is $S = \{\omega_2, \omega_6, \omega_7, \omega_8\}$.

Corollary 2. Generators of $G_p^{\Gamma_1}$ are not unique.

Proposition 3. The Γ_1 -non deranged permutation group is a cyclic group.

3.1 Undirected Cayley graph of Γ_1 -non deranged permutation group

In this section we state the main results on the undirected Cayley graph of Γ_1 -non deranged permutation groups

Proposition 4. Suppose $Cay(G_p^{\Gamma_1}, S)$ is a Cayley graph of $G_p^{\Gamma_1}$, where S is the generating set, then

$$|E(Cay(G_p^{\Gamma_1}, S))| = |G_p^{\Gamma_1}| |S|$$

proof. Suppose $S = \{\omega_{i_1}, \omega_{i_2}, \dots, \omega_{k_n}\}$ such that n is the cardinality of S . Then the following lists the set of edges of $Cay(G_p^{\Gamma_1}, S)$

$$\begin{aligned}
 &(\omega_1, \omega_1 \circ \omega_{i_1}), (\omega_2, \omega_2 \circ \omega_{i_1}), \dots, (\omega_{p-1}, \omega_{p-1} \circ \omega_{i_1}) \\
 &(\omega_1, \omega_1 \circ \omega_{j_1}), (\omega_2, \omega_2 \circ \omega_{j_1}), \dots, (\omega_{p-1}, \omega_{p-1} \circ \omega_{j_1}) \\
 &\quad \vdots \\
 &(\omega_1, \omega_1 \circ \omega_{k_n}), (\omega_2, \omega_2 \circ \omega_{k_n}), \dots, (\omega_{p-1}, \omega_{p-1} \circ \omega_{k_n})
 \end{aligned}$$

And the dimension of the array is $n \times p - 1$ which implies

$$|E(Cay(G_p^{\Gamma_1}, S))| = |G_p^{\Gamma_1}| |S|$$

And thus this completes the proof. ■

The figure below shows a Cayley graph of $G_5^{F_1}$

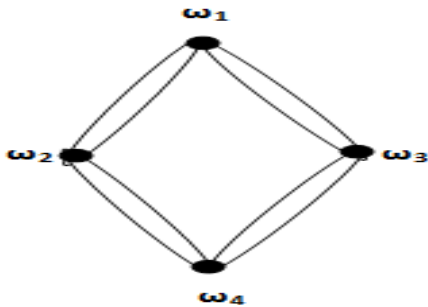


Figure 1: The Cayley graph of $G_5^{F_1}$

We know that $|G_5^{F_1}| = 4$ and from self multiplication table of $p = 5$, $G_5^{F_1}$ has only ω_2 and ω_3 as set of generators, so $|S| = 2$, therefore, $|E(Cay(G_5^{F_1}, S))| = |G_5^{F_1}| |S| = (4)(2) = 8$.

Proposition 5. Suppose $(Cay(G_p^{F_1}, S))$ is a Cayley graph of $G_p^{F_1}$, where S is the generating set of $G_p^{F_1}$ then,

$$|V(Cay(G_p^{F_1}, S))| = p - 1$$

Proof: Ibrahim et al. (2022c). The cardinality of $G_p^{F_1}$ is $p - 1$. Since the vertex set of $Cay(G_p^{F_1}, S)$ contains all the elements of $G_p^{F_1}$; it implies that $|V(Cay(G_p^{F_1}, S))| = p - 1$ ■

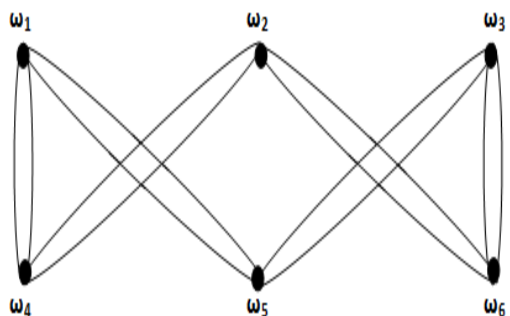


Figure 2: The Cayley graph of $G_7^{F_1}$. $|V(Cay(G_7^{F_1}, S))| = p - 1 = 7 - 1$

Corollary 3. The Cayley graph $Cay(G_p^{F_1}, S)$ of the F_1 -non deranged permutation group is of order $p - 1$.

Remark The vertex set of the Cayley graph $Cay(G_p^{F_1}, S)$ of F_1 -non deranged permutation is even.

Proposition 6. Suppose S is the generating group set of the F_1 -non deranged permutation group, then it's Cayley graph $Cay(G_p^{F_1}, S)$ is a $((p - 1); (p - 1)|S|)$ graph.

proof. The proof follows from Corollary 4.2 and Corollary 4.4 ■

Proposition 7. Let $Cay(G_p^{F_1}, S)$ be Cayley graph of $G_p^{F_1}$ where S is the set of generator and V be set of vertices of $Cay(G_p^{F_1}, S)$, then

$$\sum_{v \in V} deg(v) = 2((p - 1)|S|) \quad (1)$$

proof. By proposition 4.1,

$$|E(Cay(G_p^{F_1}, S))| = |G_p^{F_1}| |S|$$

from hand shaking lemma, equation (1) becomes

$$\sum_{v \in V} deg(v) = 2E(Cay(G_p^{F_1}, S)) = 2((p - 1)|E|)$$

And thus this complete the proof. ■

Proposition 8. Suppose S is the set of generators of $G_p^{F_1}$: Then the Cayley graph $Cay(G_p^{F_1}, S)$ graphs is a multigraph.

proof. By the definition of a Cayley graph two vertices $\omega_i, \omega_j \in G_p^{F_1}$ are connected by an edge (ω_i, ω_j) if $\omega_i \circ \omega_k = \omega_j$ for $\omega_k \in S$. Since $G_p^{F_1}$ is commutative, if (ω_i, ω_j) is an edge in $E(Cay(G_p^{F_1}, S))$ then (ω_j, ω_i) is an edge in $E(Cay(G_p^{F_1}, S))$ Therefore $Cay(G_p^{F_1}, S)$ is a multigraph.

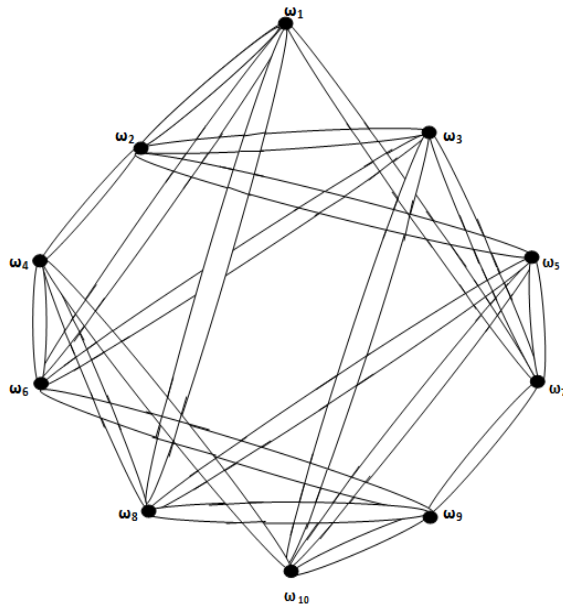


Figure 3: (The Cayley graph of $G_{11}^{\Gamma_1}$)
The figure above is a multigraph as it has multiple edges.

Corollary 4. Suppose $\text{Cay}(G_p^{\Gamma_1}, S)$ is a Cayley graph of $G_p^{\Gamma_1}$ and S is the generating set of $G_p^{\Gamma_1}$. If $|S|$ is even then for every $(\omega_i, \omega_j) \in E(\text{Cay}(G_p^{\Gamma_1}, S))$, there exist $(\omega_j, \omega_i) \in E(\text{Cay}(G_p^{\Gamma_1}, S))$.

Remark. We note that if $|S|$ is odd then there exist some $(\omega_i, \omega_j) \in E(\text{Cay}(G_p^{\Gamma_1}, S))$, such that $(\omega_j, \omega_i) \notin E(\text{Cay}(G_p^{\Gamma_1}, S))$.

4.0. Conclusion

In conclusion, we observed that not all element of Γ_1 -non deranged permutation group generate the elements of the group therefore the generating set is a proper subset of $G_p^{\Gamma_1}$ and are not unique and also the undirected Cayley graphs of Γ_1 -non deranged permutation group is not simple.

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