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## The Transform-Transformer Approach: Unveiling the Odd Transmuted Rayleigh-X Family of Distributions

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**ABSTRACT**

The paper presents a novel class (family) of statistical distributions termed Odd Transmuted Rayleigh-X (OTR-X) that was created through a transform-transformer (T-X) approach. The CDF and PDF of the OTR-X family were derived. The available statistical literature studied earlier highlighted that almost all generalized distributions (in which one or more parameters were added) performed well and have better presentation of data than their counterparts with less number of parameters. This has motivated us to developed new family that is capable of producing new distributions. The research paper also presented a clear mathematical formula for several characteristics of the OTR-X family, such as the ordinary moments, moment generating, quantile, and reliability function. In order to find the estimate of the corresponding parameters of the OTR-X family, the technique of maximum likelihood is used in the study. A new sub-model Odd Transmuted Rayleigh Inversed Exponential Distribution (OTRIED) was generated from the OTR-X class and compared its performance to Transmuted Inversed Exponential Distribution (TIED), Exponential Inversed Exponential Distribution (EIED), and Inversed Exponential Distribution using two different datasets. The results have shown that the proposed distribution out performed its competitors when using two different real-world datasets. Furthermore, the proposed distribution can be practicalized to any skewed dataset.

**Keywords:** Transmuted Rayleigh, moments, entropy measures, quantile function, and transform- transformer

**1.0 INTRODUCTION**

There is significant academic concentration in both the theoretical and practical aspects of statistical distributions. As part of this interest, many new distributions are being developed to better represent data collections and facilitate parametric statistical inferences and modeling. It is generally better to use an appropriate probability distribution to represent a data set, rather than transforming it, as transformation may alter the data's original characteristics. This view point is supported by various sources, including Merovci (2013), Usman et al. (2019), Sen et al. (2019), Zakari and Usman (2020), Usman et al. (2021), and Musa et al. (2021).

Statistical distributions play a vital role in making parametric inferences and modeling real-world phenomena. Researchers have developed different techniques for creating compound distributions as a result. Some early methods for

creating new families of continuous distributions included using Pearson's differential equations

(1895), Johnson's translation (1949), and Tukey's quantile functions (1960). In recent years, the idea of creating new families of distributions through various means has remained prevalent, as there is a need for more flexible distributions. Lee et al. (2013) reported that since the 1980s, combining different distributions has become one of the most commonly used techniques for creating new families of distributions. This approach involves merging two or more distributions or extending the distribution by a parameter(s). The resulting new family is then used to better model and represent data.

Recent studies in statistics have revealed that the methods for generating new probability distributions are commonly referred to as generators, families, or generalized classes of probability distributions. These techniques result

in the creation of new distributions that extend the scope of already existing distributions and provide greater flexibility in modeling real-world data sets. In some cases, the use of expanded distributions has resulted in better parametric fits for certain datasets. Usman et al. (2020) came up with generator that can be used to create different forms of distributions which can better fit real-world datasets with different shapes.

Numerous generators and distributions have been studied. Examples includes; The Alpha Power Transformation Family developed by mead et al. (2019), Exponentiated Transmuted Inverse Exponential Distribution by Mohammed and Yahaya (2019), A new generalized family of odd Lindley-G distributions presented by Chipepa et al. (2019), A new extended two parameter distribution by Al-Mofleh et al. (2020), The Weibull Marshall-Olkin lindley distribution by Afify et al. (2020), The Topp-Leone-Marshall-Olkin-G family of distributions by Chipepa et al. (2020), A new generator for proposing flexible lifetime distributions by Aslam et al. (2020), A new Kumaraswamy generalized family of distributions proposed by Tahir et al. (2020), A new generalization of power function distribution presented by Hassan and Nassr (2020), Odd generalized exponential power function distribution produced by Hassan (2021), A new class of power function distribution developed by Hassan and Assar (2021), Zero-truncated Poisson-power function distribution by Okorie et al. (2021), NBeta-G family of distributions introduced by Alsolmi (2023), Exponentiated odd Lindley-X family presented by Abd El-Bar and Maria do Carmo (2023), A new extension of the exponentiated generalized-G family of distributions by Keganne (2023), A novel logarithmic approach to generate new probability distributions by Zhao et al. (2023) and The Gamma Odd Weibull Generalized-G Family of Distributions by Oluyede1 and Warahena-Liyanage (2023). Additionally, other G-families include; Mohammed and Ugwuowo (2020), Usman et al. (2021), Yahaya and Doguwa (2021), and many more.

Alzaatreh et al. (2013) established a method of generating probability (compound) distributions termed as Transform-transformer (T-X) method. The Transform component of the new family in this research is the Transmuted Rayleigh distribution, while the Transformer component is an odd function of any continuous baseline distribution selected arbitrarily.

This research is designed in the following ways: In section 2, the CDF and PDF of the constructed OTR-X family are provided. Section 3 presents the expansion of the CDF and PDF. In section 4, various statistical features such as moment, moment generating as well as quantile function, entropy, distribution of order statistics, and reliability are obtained. Section 5 discuss the estimation of distribution parameters, while section 6 presents a sub-model of the OTR-X. Furthermore, Section 7 applies the proposed distributions to a real dataset. Finally, the paper concludes in section 8.

## 2.0 DEFINITION OF THE TRANSMUTED RAYLEIGH DISTRIBUTION

Merovci (2013) provided (1) and (2) as the distribution and density function of a random variable (RV)  $T$  with the transmuted Rayleigh model:

$$R(t) = \left(1 - e^{-\frac{t^2}{2\sigma^2}}\right) \left(1 + \lambda e^{-\frac{t^2}{2\sigma^2}}\right) \quad (1)$$

$$r(t) = \frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}} \left(1 - \lambda + 2\lambda e^{-\frac{t^2}{2\sigma^2}}\right) \quad (2)$$

with  $\sigma > 0, \lambda > 0$  and  $t > 0$

### 2.1 The Odd Transmuted Rayleigh-X (OTR-X) Family of Distribution

Referring to the work of one of the notable authors in the area of proposing new distributions by name Alzaatreh et al. (2013) who proposed a definition for obtaining cumulative distribution function (CDF) that lead to the generation of novel class of distributions. The rationale behind adopting T-X method is to pave way for producing distributions with varying shapes.

In this paper, we referred to the new generated class (family) as the Odd Transmuted Rayleigh-X (OTR-X), is defined as a function of the CDF of an arbitrarily chosen continuous baseline

distribution and a transform function, where the transform function is the Transmuted Rayleigh distribution.

Let a RV  $X$  have the following CDF;

$$F(x) = \int_a^{W(G(x))} r(t) dt \tag{3}$$

From (3), the link function denoted by  $W(G(x))$  is considered to be a mathematical function of a baseline distribution denoted by  $G(x)$  of any RV  $X$ , and  $r(t)$  is the transform distribution. For detail explanations of this method one can refer to Alzaatreh et al. (2013).

The OTR-X family was obtained in equation (4) by replacing  $r(t)$  in (3) with the PDF  $r(t)$  in (2), since  $t > 0$ , the chosen link function  $W(G(x))$  is  $\frac{G(x)}{1-G(x)} = \frac{G(x)}{\bar{G}(x)}$ .

The OTR-X family's distribution function is obtained as;

$$F(x, \sigma, \lambda, \phi) = \left( 1 - e^{-\frac{1}{2\sigma^2} \left( \frac{G(x;\phi)}{\bar{G}(x;\phi)} \right)^2} \right) \left( 1 + \lambda e^{-\frac{1}{2\sigma^2} \left( \frac{G(x;\phi)}{\bar{G}(x;\phi)} \right)^2} \right) \tag{4}$$

The PDF is derived by applying the concept of differentiation method and is obtained as;

$$f(x, \sigma, \lambda, \phi) = \sigma^{-2} g(x; \phi) \frac{G(x; \phi)}{(\bar{G}(x; \phi))^3} e^{-\frac{1}{2\sigma^2} \left( \frac{G(x; \phi)}{\bar{G}(x; \phi)} \right)^2} \left[ 1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2} \left( \frac{G(x; \phi)}{\bar{G}(x; \phi)} \right)^2} \right] \tag{5}$$

The OTR-X family of distributions can now be used for generating new compound distributions for any continuous distribution as a baseline.

### 2.2 Checking the Validity of OTR-X Family of Distribution

To ensure the correctness of the proposed probability distribution function (PDF) of a new family, it is common practice to verify it using a proposition that can be proven.

**Proposition 1:** It is possible to define a probability density function for the odd transmuted Rayleigh-X family of distributions.

Proof; using equation (5), the total probability must be equal 1, so we have;

$$\int_0^{\infty} f(x, \sigma, \lambda, \phi) dx = \int_0^{\infty} \frac{g(x; \phi)}{\sigma^2} \frac{G(x; \phi)}{(\bar{G}(x; \phi))^3} e^{-\frac{1}{2\sigma^2} \left( \frac{G(x; \phi)}{\bar{G}(x; \phi)} \right)^2} \left[ 1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2} \left( \frac{G(x; \phi)}{\bar{G}(x; \phi)} \right)^2} \right] dx = 1 \tag{6}$$

the integral can be solved using the substitution method

$$\text{let } y = e^{-\frac{1}{2\sigma^2} \left( \frac{G(x; \phi)}{\bar{G}(x; \phi)} \right)^2},$$

which means as;  $y = 1$ ,  $x = 0$  and  $y = 0$ ,  $x = \infty$

$$\frac{dy}{dx} = -\frac{g(x; \phi)G(x; \phi)}{\sigma^2(1-G(x; \phi))^3} e^{-\frac{1}{2\sigma^2} \left( \frac{G(x; \phi)}{\bar{G}(x; \phi)} \right)^2} \tag{and}$$

$$dx = \frac{-\sigma^2 (1-G(x; \phi))^3 dy}{g(x; \phi)G(x; \phi) e^{-\frac{1}{2\sigma^2} \left( \frac{G(x; \phi)}{\bar{G}(x; \phi)} \right)^2}}$$

substituting the value of  $y$  and  $dx$  in equation (6), then;

$$\int_0^{\infty} f(x, \sigma, \lambda, \phi) dx = \int_0^1 \sigma^{-2} g(x; \phi) \frac{G(x; \phi)}{(\bar{G}(x; \phi))^3} e^{-\frac{1}{2\sigma^2} \left( \frac{G(x; \phi)}{\bar{G}(x; \phi)} \right)^2} [1 - \lambda + 2\lambda y] \frac{\sigma^2 (\bar{G}(x; \phi))^3 dy}{g(x; \phi)G(x; \phi) e^{-\frac{1}{2\sigma^2} \left( \frac{G(x; \phi)}{\bar{G}(x; \phi)} \right)^2}}$$

$$\int_0^{\infty} f(x, \sigma, \lambda, \phi) dx = \int_0^1 [1 - \lambda + 2\lambda y] dy = [y]_0^1 - \lambda [y]_0^1 + 2\lambda \left[ \frac{y^2}{2} \right]_0^1 = 1 - \lambda + \lambda = 1$$

### 2.3 Limiting Behaviour of the CDF and PDF

**Lemma 1:** The limit behaviour of the density function of the OTR-X class is zero as  $x$  tends to

both upper and lower limit. This can be obtained by taking the limit of (5) as  $x \rightarrow \infty$  and  $x \rightarrow 0$ . This implies, as  $x \rightarrow \infty$ ,  $g(x; \phi) \rightarrow 0$

$$\lim_{x \rightarrow \infty} f(x, \sigma, \lambda, \phi) = \left( \frac{0}{\sigma^2} \frac{G(x; \phi)}{\overline{G}(x; \phi)^3} e^{-\frac{1}{2\sigma^2} \left( \frac{G(x; \phi)}{\overline{G}(x; \phi)} \right)^2} \left[ 1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2} \left( \frac{G(x; \phi)}{\overline{G}(x; \phi)} \right)^2} \right] \right) = 0$$

This implies as  $x \rightarrow 0$ ,  $g(x; \phi) \rightarrow 0$

$$\lim_{x \rightarrow 0} f(x, \sigma, \lambda, \phi) = \left( \frac{0}{\sigma^2} \frac{G(x; \phi)}{\overline{G}(x; \phi)^3} e^{-\frac{1}{2\sigma^2} \left( \frac{G(x; \phi)}{\overline{G}(x; \phi)} \right)^2} \left[ 1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2} \left( \frac{G(x; \phi)}{\overline{G}(x; \phi)} \right)^2} \right] \right) = 0$$

The evidence presented above suggests that a mode can exist in a distribution generated from the OTR-X family of distributions.

**Lemma 2.** The limit behaviour of the distribution function of the OTR-X class is one and zero as  $x$  tends to upper and lower limit respectively. This can be obtained by considering the limit of the (4) as  $x \rightarrow \infty$  and  $x \rightarrow 0$ .

That is, as  $x$  approaches infinity, the baseline distribution function  $G(x; \phi) \rightarrow 1$

$$\lim_{x \rightarrow \infty} F(x, \sigma, \lambda, \phi) = \left( (\lambda - 1) e^{-\frac{1}{2\sigma^2} \left( \frac{1}{1-1} \right)^2} - \lambda e^{-\frac{1}{\sigma^2} \left( \frac{1}{1-1} \right)^2} + 1 \right) = 1$$

as  $x$  approaches zero, the baseline distribution function

$$G(x; \phi) \rightarrow 0$$

$$\lim_{x \rightarrow 0} F(x, \sigma, \lambda, \phi) = \left( (\lambda - 1) e^{-\frac{1}{2\sigma^2} \left( \frac{0}{1-0} \right)^2} - \lambda e^{-\frac{1}{\sigma^2} \left( \frac{0}{1-0} \right)^2} + 1 \right) = 0$$

### 3.0 EXPANSION OF OTR-X CLASS'S CDF AND PDF

The proposed family's CDF can be written as;

$$F(x, \sigma, \lambda, \phi) = (\lambda - 1) e^{-\frac{1}{2\sigma^2} \left( \frac{G(x; \phi)}{\overline{G}(x; \phi)} \right)^2} - \lambda e^{-\frac{1}{\sigma^2} \left( \frac{G(x; \phi)}{\overline{G}(x; \phi)} \right)^2} + 1$$

$$\text{let, } a = e^{-\frac{1}{2\sigma^2} \left( \frac{G(x; \phi)}{\overline{G}(x; \phi)} \right)^2} \text{ and } b = e^{-\frac{1}{\sigma^2} \left( \frac{G(x; \phi)}{\overline{G}(x; \phi)} \right)^2} \tag{7}$$

using Maclaurin's series expansion for the a and b

$$a = \sum_{i=0}^{\infty} \frac{(-1)^i \left( (2\sigma^2)^{-1} \left( \frac{G(x; \phi)}{\overline{G}(x; \phi)} \right)^2 \right)^i}{i!}$$

$$b = \sum_{i=0}^{\infty} \frac{(-1)^i (2\sigma^2)^{-i} \left( \frac{G(x; \phi)}{\overline{G}(x; \phi)} \right)^{2i}}{i!}$$

$$b = \sum_{i=0}^{\infty} \frac{(-1)^i \left( \frac{1}{\sigma^2} \left( \frac{G(x; \phi)}{\overline{G}(x; \phi)} \right)^2 \right)^i}{i!}$$

$$b = \sum_{i=0}^{\infty} \frac{(-1)^i \left( \frac{1}{\sigma^2} \right)^i \left( \frac{G(x; \phi)}{\overline{G}(x; \phi)} \right)^{2i}}{i!}$$

substituting for a and b in the CDF (7) we have;

$$F(x, \sigma, \lambda, \phi) = (\lambda - 1) \sum_{i=0}^{\infty} \frac{(-1)^i (2\sigma^2)^{-i}}{i!} \frac{G(x; \phi)^{2i}}{\overline{G}(x; \phi)^{2i}} - \lambda \sum_{i=0}^{\infty} \frac{(-1)^i (\sigma^2)^{-i}}{i!} \frac{G(x; \phi)^{2i}}{\overline{G}(x; \phi)^{2i}} + 1 \tag{8}$$

Let,

$$c = (1 - G(x; \phi))^{-2i}$$

using Binomial expansion for c;

$$c = (1 - G(x; \phi))^{-2i} = \sum_{j=0}^{\infty} (-1)^j \binom{-2i}{j} G(x; \phi)^j$$

substituting the value of c in equation (8) yields;

$$\begin{aligned}
 &= 1 + (\lambda - 1) \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j} \left(\frac{1}{2\sigma^2}\right)^i}{i!} \binom{-2i}{j} G(x; \phi)^{2i+j} \\
 &\quad - \lambda \sum_{i=0}^{\infty} \frac{(-1)^{i+j} \left(\frac{1}{\sigma^2}\right)^i}{i!} \binom{-2i}{j} G(x; \phi)^{2i+j} \\
 &= \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{i!} \binom{-2i}{j} \left( (\lambda - 1) \left(\frac{1}{2\sigma^2}\right)^i - \lambda \left(\frac{1}{\sigma^2}\right)^i \right) \\
 &\quad G(x; \phi)^{2i+j} + 1 \\
 F(x, \sigma, \lambda, \phi) &= 1 + \sum_{i,j=0}^{\infty} m_{i,j} G(x; \phi)^\alpha \\
 &= 1 + \sum_{i,j=0}^{\infty} m_{i,j} \Pi_\alpha(x) \tag{9}
 \end{aligned}$$

where

$$m_{i,j} = \frac{(-1)^{i+j}}{i!} \binom{-2i}{j} \left( (\lambda - 1) \left(\frac{1}{2\sigma^2}\right)^i - \lambda \left(\frac{1}{\sigma^2}\right)^i \right)$$

$\alpha = 2i + j$  and  $\Pi_\alpha(x) = G(x; \phi)^\alpha$  corresponded to the CDF of the well-known class (Exponentiated-G) with parameter  $\alpha$ .

Differentiating (9) with regards to  $x$ , the density is derived as;

$$\begin{aligned}
 f(x, \sigma, \lambda, \phi) &= \sum_{i,j=0}^{\infty} m_{i,j} \alpha g(x; \phi) G(x; \phi)^{\alpha-1} \\
 &= \sum_{i,j=0}^{\infty} m_{i,j} \pi_\alpha(x)
 \end{aligned}$$

(10)

Where  $\pi_\alpha(x) = \alpha g(x; \phi) G(x; \phi)^{\alpha-1}$  stands for the density of Exp-G class.

#### 4.0 STATISTICAL PROPERTIES OF OTR-X FAMILY OF DISTRIBUTION

When introducing new families of distributions, it is customary to analyse certain statistical characteristics. In this study, the following features were considered.

#### 4.1 Moments of the OTR-X family

The moment of the OTR-X class of distributions is define by the following;

$$E(X^r) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} m_{i,j} E(X_{i,j}^r) \tag{11}$$

Where,  $X_{i,j}$  represents the exp-G class with  $2i + j$  as parameter.

#### 4.2 Incomplete Moments of the proposed family

The  $r^{th}$  incomplete moment of the OTR-X class of distributions is obtained as;

$$I_r(y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} m_{i,j} I_{i,j}(y) \tag{12}$$

Where,

$$I_{i,j}(y) = \int_0^y x^r \pi_\alpha(x) dx$$

#### 4.3 Moment Generating Function of the family

The MGF of the OTR-X class is given as;

$$M_X(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} m_{i,j} E\left(e^{tX_{i,j}}\right) \tag{13}$$

$$\text{where, } E(e^{tX_{i,j}}) = \int_0^{\infty} e^{tX_{i,j}} \pi_\alpha(x) dx$$

#### 4.4 Quantile Function of the OTR-X family

##### Proposition 2

The derived quantile function (qf) of the OTR-X class is presented as;

$$\begin{aligned}
 Q_x(u) &= G^{-1} \\
 &\left( \frac{\left( \left( 2\sigma^2 \left( -\log \left( \frac{(\lambda - 1) \pm \sqrt{(\lambda - 1)^2 + (4\lambda(1-u))}}{2\lambda} \right) \right) \right)^{\frac{1}{2}} \right)}{1 + \left( 2\sigma^2 \left( -\log \left( \frac{(\lambda - 1) \pm \sqrt{(\lambda - 1)^2 + (4\lambda(1-u))}}{2\lambda} \right) \right) \right)^{\frac{1}{2}}} \right) \tag{14}
 \end{aligned}$$

proof;

By definition; let a RV  $X$  follows OTR-X class, with a function given in (5) then the quantile function denoted by  $Q_x(u)$  for  $0 < u < 1$  can be derived by this relation:

$$Q_x(u) = F^{-1}(u) \tag{15}$$

by utilizing the CDF in equation (7), we can let

$$y = e^{-\frac{1}{2\sigma^2} \left( \frac{G(x;\phi)}{G(x;\phi)} \right)^2}$$

So, equation (7) can be written as;

$$F(y) = 1 + (\lambda - 1)y - \lambda y^2.$$

Using equation (15) we can have;

$$\begin{aligned} Q_y(u) &= (F(y))^{-1} \quad \text{for } 0 < u < 1 \\ u &= 1 + (\lambda - 1)y - \lambda y^2 \\ -\lambda y^2 + (\lambda - 1)y + 1 - u &= 0 \end{aligned} \tag{16}$$

By solving equation (16) we obtain;

$$y = \frac{(\lambda - 1) \pm \sqrt{(\lambda - 1)^2 + (4\lambda(1 - u))}}{2\lambda}$$

from  $y = e^{-\frac{1}{2\sigma^2} \left( \frac{G(x;\phi)}{G(x;\phi)} \right)^2}$

we get  $G(x;\phi) = \frac{(2\sigma^2(-\log(y)))^{\frac{1}{2}}}{(1 + (2\sigma^2(-\log(y)))^{\frac{1}{2}})}$

substitute the value of y in  $G(x;\phi)$ ,  $Q_x(u)$  be becomes;

$$Q_x(u) = G^{-1} \left( \frac{\left( 2\sigma^2 \left( -\log \left( \frac{(\lambda - 1) \pm \sqrt{(\lambda - 1)^2 + (4\lambda(1 - u))}}{2\lambda} \right) \right) \right)^{\frac{1}{2}}}{1 + \left( 2\sigma^2 \left( -\log \left( \frac{(\lambda - 1) \pm \sqrt{(\lambda - 1)^2 + (4\lambda(1 - u))}}{2\lambda} \right) \right) \right)^{\frac{1}{2}}} \right) \tag{17}$$

Where  $G^{-1}(\cdot)$  represent the qf of the parent model.

Now, equation (17) is the qf of the OTR-X family of distribution, by setting  $u = \frac{1}{4}$ ,  $u = \frac{1}{2}$  and

$u = \frac{3}{4}$  in it, the first, the median (second) and third quartile of the family are derived respectively as;

$$Q_x(0.25) = G^{-1} \left( \frac{\left( 2\sigma^2 \left( -\log \left( \frac{(\lambda - 1) \pm \sqrt{\lambda^2 + \lambda + 1}}{2\lambda} \right) \right) \right)^{\frac{1}{2}}}{1 + \left( 2\sigma^2 \left( -\log \left( \frac{(\lambda - 1) \pm \sqrt{\lambda^2 + \lambda + 1}}{2\lambda} \right) \right) \right)^{\frac{1}{2}}} \right) \tag{18}$$

$$Q_x(0.5) = G^{-1}$$

$$\left( \frac{\left( 2\sigma^2 \left( -\log \left( \frac{(\lambda - 1) \pm \sqrt{\lambda^2 + 1}}{2\lambda} \right) \right) \right)^{\frac{1}{2}}}{1 + \left( 2\sigma^2 \left( -\log \left( \frac{(\lambda - 1) \pm \sqrt{\lambda^2 + 1}}{2\lambda} \right) \right) \right)^{\frac{1}{2}}} \right) \tag{19}$$

$$Q_x(0.75) = G^{-1}$$

$$\left( \frac{\left( 2\sigma^2 \left( -\log \left( \frac{(\lambda - 1) \pm \sqrt{\lambda^2 - \lambda + 1}}{2\lambda} \right) \right) \right)^{\frac{1}{2}}}{1 + \left( 2\sigma^2 \left( -\log \left( \frac{(\lambda - 1) \pm \sqrt{\lambda^2 - \lambda + 1}}{2\lambda} \right) \right) \right)^{\frac{1}{2}}} \right) \tag{20}$$

### 4.5 Entropy Measures

If a R.V X, follows the OTR-X, then the Renyi entropy and the q-entropy are respectively defined as;

$$I_{\theta}(x) = \frac{1}{1-\theta} \log \left\{ \left( \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} m_{i,j} \right)^{\theta} \int_0^{\infty} (\pi_{\alpha}(x))^{\theta} dx \right\};$$

$\theta > 0$  and  $\theta \neq 1$

(21)

and

$$H_q(x) = \frac{1}{q-1} \log \left\{ 1 - \left( \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} m_{i,j} \right)^q \int_0^{\infty} (\pi_{\alpha}(x))^q dx \right\};$$

$q > 0$  and  $q \neq 1$

(22)

#### 4.6 Distribution of Order Statistics

The density of the  $p^{th}$  order statistic (OS)  $X_{p:n}$  are generated as;

$$f_{p:n}(x) = \frac{n!}{(p-1)!(n-p)!} f(x; \sigma, \lambda, \phi) F(x; \sigma, \lambda, \phi)^{p-1}$$

$$[1 - F(x; \sigma, \lambda, \phi)]^{n-p}$$

(23)

But

$$[1 - F(x; \sigma, \lambda, \phi)]^{n-p} =$$

$$\sum_{k=0}^{n-p} (-1)^k \binom{n-p}{k} [F(x; \sigma, \lambda, \phi)]^k$$

(24)

and

$$\binom{n-p}{k} = \frac{(n-p)!}{(n-p-k)!k!}$$

(25)

substituting equation (24) and (25) in (23), we have

$$f_{p:n}(x) = \sum_{k=0}^{n-p} \frac{(-1)^k n!}{(p-1)!(n-p-k)!k!} f(x; \sigma, \lambda, \phi) [F(x; \sigma, \lambda, \phi)]^{p+k-1}$$

(26)

By replacing (10) and (9) in (26) we have the result as;

$$f_{p:n}(x) = \sum_{k=0}^{n-p} \frac{(-1)^k n!}{(p-1)!(n-p-k)!k!} \left( \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} m_{i,j} \pi_{\alpha}(x) \right)$$

$$\left( 1 + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} m_{i,j} \Pi_{\alpha}(x) \right)^{p+k-1}$$

(27)

Hence, by substituting  $p=1$  and  $p=n$  in the PDF (23), we have the minimum and maximum OS of the OTR-X class of distribution, which are respectively derived as;

$$f_{1:n}(x) = \frac{n!}{(n-1)!} \left( \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} m_{i,j} \pi_{\alpha}(x) \right)$$

$$\left( - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} m_{i,j} \Pi_{\alpha}(x) \right)^{n-1}$$

(28)

and,

$$f_{n:n} = \frac{n!}{(n-1)!} \left( \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} m_{i,j} \pi_{\alpha}(x) \right) \left( 1 + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} m_{i,j} \Pi_{\alpha}(x) \right)^{n-1}$$

(29)

#### 4.7 Survival and Hazard Functions

The OTR-X family has a survival function as;

$$S(x, \sigma, \lambda, \phi) = 1 - F(x, \sigma, \lambda, \phi)$$

$$= e^{-\frac{1}{2\sigma^2} \left( \frac{G(x;\phi)}{G(x;\phi)} \right)^2} \left( 1 - \lambda + \lambda e^{-\frac{1}{2\sigma^2} \left( \frac{G(x;\phi)}{G(x;\phi)} \right)^2} \right)$$

(30)

The proposed family has the hazard function as;

$$h(x, \sigma, \lambda, \phi) = \frac{f(x, \sigma, \lambda, \phi)}{S(x, \sigma, \lambda, \phi)}$$

$$= \frac{\frac{g(x;\phi)}{\sigma^2} \frac{G(x;\phi)}{(G(x;\phi))^3} e^{-\frac{1}{2\sigma^2} \left( \frac{G(x;\phi)}{G(x;\phi)} \right)^2} \left[ 1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2} \left( \frac{G(x;\phi)}{G(x;\phi)} \right)^2} \right]}{e^{-\frac{1}{2\sigma^2} \left( \frac{G(x;\phi)}{G(x;\phi)} \right)^2} \left[ 1 - \lambda + \lambda e^{-\frac{1}{2\sigma^2} \left( \frac{G(x;\phi)}{G(x;\phi)} \right)^2} \right]}$$

(31)

**5.Estimation of Parameters of the OTR-X Class of Distributions**

In this context, the suggested family’s parameters are computed using the MLE method. Suppose  $X_1, X_2, \dots, X_n$  consist of  $n$  random variables drawn from a population with an OTR-X family of distribution, where  $\Theta = (\sigma, \lambda, \phi)^T$  is a vector with an unknown parameter, and sample values  $x_1, x_2, \dots, x_n$  have a joint density function of  $f(x_1, x_2, \dots, x_n; \Theta)$ . Then the likelihood function  $L(\Theta)$ , of the random samples is given as:

$$L(\Theta) = \prod_{i=1}^n f(x_i; \sigma, \lambda, \phi) \tag{32}$$

$$= \prod_{i=1}^n \left\{ \frac{g(x_i; \phi)}{\sigma^2} \frac{G(x_i; \phi)}{(\bar{G}(x_i; \phi))^3} e^{-\frac{1}{2\sigma^2} \left( \frac{G(x_i; \phi)}{\bar{G}(x_i; \phi)} \right)^2} \left[ 1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2} \left( \frac{G(x_i; \phi)}{\bar{G}(x_i; \phi)} \right)^2} \right] \right\}$$

taking the natural log of  $L(\Theta)$ , gives the log-likelihood denoted by  $\ell$

$$\ell = -2n \log \sigma + \sum_{i=1}^n \log g(x_i; \phi) + \sum_{i=1}^n \log G(x_i; \phi) - 3 \sum_{i=1}^n \log \bar{G}(x_i; \phi) - \frac{1}{2\sigma^2} \sum_{i=1}^n \left( \frac{G(x_i; \phi)}{\bar{G}(x_i; \phi)} \right)^2 + \sum_{i=1}^n \log \left( 1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2} \left( \frac{G(x_i; \phi)}{\bar{G}(x_i; \phi)} \right)^2} \right)$$

let,

$$ci = \left( \frac{G(x_i; \phi)}{\bar{G}(x_i; \phi)} \right)^2$$

$$\ell = \left\{ \begin{aligned} & -2n \log \sigma + \sum_{i=1}^n \log g(x_i; \phi) + \sum_{i=1}^n \log G(x_i; \phi) - 3 \sum_{i=1}^n \log \bar{G}(x_i; \phi) \\ & - \frac{1}{2\sigma^2} \sum_{i=1}^n ci + \sum_{i=1}^n \log \left( 1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2} ci} \right) \end{aligned} \right\} \tag{33}$$

The  $U(\Theta)$  is obtained by performing partial differentiation for  $\ell$  with regards to the individual parameters and equating the answer to 0.

That is,

$$U(\Theta) = \frac{\partial \ell}{\partial \Theta} = \left( \frac{\partial \ell}{\partial \phi}, \frac{\partial \ell}{\partial \phi}, \frac{\partial \ell}{\partial \phi} \right)^T$$

$$U_\lambda = \frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^n \frac{2e^{-\frac{1}{2\sigma^2} ci} - 1}{1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2} ci}} \tag{34}$$

$$U_\sigma = \frac{\partial \ell}{\partial \sigma} = \frac{-2n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^n ci + \frac{2\lambda e^{-\frac{1}{2\sigma^2} ci} ci \sigma^{-3}}{1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2} ci}} \tag{35}$$

$$U_\phi = \frac{\partial \ell}{\partial \phi} = \left\{ \begin{aligned} & \sum_{i=1}^n \frac{g(x_i; \phi)}{g(x_i; \phi)} + \sum_{i=1}^n \frac{g(x_i; \phi)}{G(x_i; \phi)} + 3 \sum_{i=1}^n \frac{g(x_i; \phi)}{G(x_i; \phi)} \\ & - \sum_{i=1}^n \frac{1}{\sigma^2} \frac{g(x_i; \phi) G(x_i; \phi)}{\bar{G}(x_i; \phi)^3} - \sum_{i=1}^n \frac{2\lambda e^{-\frac{1}{2\sigma^2} ci} \frac{g(x_i; \phi) G(x_i; \phi)}{\bar{G}(x_i; \phi)^3}}{1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2} ci}} \end{aligned} \right\} \tag{36}$$

**6. A MODEL OBTAINED FROM OTR-X CLASS OF DISTRIBUTION**

Assume that the parent model is Inverse Exponential (IE) distribution with the following functions given as:

$$F(x; \eta) = e^{-\frac{\eta}{x}}$$

(37)

and

$$f(x; \eta) = \frac{\eta}{x^2} e^{-\frac{\eta}{x}}$$

(38)

with  $x > 0$  and  $\eta > 0$

The resulting model called Odd Transmuted Rayleigh Inversed Exponential Distribution (OTRIED) was introduced with equations (39) and (40) serving as its CDF and PDF, respectively. These equations described both CDF and PDF of OTRIED.

$$F(x; \sigma, \lambda, \eta) = \left( 1 - e^{-\frac{1}{2\sigma^2} \left( \frac{\frac{\eta}{e^x} - \frac{\eta}{1-e^x}}{\frac{\eta}{1-e^x}} \right)^2} \right) \left( 1 + \lambda e^{-\frac{1}{2\sigma^2} \left( \frac{\frac{\eta}{e^x} - \frac{\eta}{1-e^x}}{\frac{\eta}{1-e^x}} \right)^2} \right)$$

$$F(x; \sigma, \lambda, \eta) = \left( 1 - e^{-\frac{1}{2\sigma^2} \left( e^x - 1 \right)^{-2}} \right) \left( 1 + \lambda e^{-\frac{1}{2\sigma^2} \left( e^x - 1 \right)^{-2}} \right)$$

(39)

We obtained (40) as the associated PDF of the OTRIED;

$$f(x; \sigma, \lambda, \eta) = \frac{\eta e^{-\frac{\eta}{x}}}{x^2 \sigma^2} \frac{e^{-\frac{\eta}{x}}}{\left( 1 - e^{-\frac{\eta}{x}} \right)^3} e^{-\frac{1}{2\sigma^2} \left( \frac{\frac{\eta}{e^x} - \frac{\eta}{1-e^x}}{\frac{\eta}{1-e^x}} \right)^2}$$

$$\left[ 1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2} \left( \frac{\frac{\eta}{e^x} - \frac{\eta}{1-e^x}}{\frac{\eta}{1-e^x}} \right)^2} \right]$$

$$f(x; \sigma, \lambda, \eta) = \frac{\eta e^{-\frac{\eta}{x}}}{x^2 \sigma^2 \left( 1 - e^{-\frac{\eta}{x}} \right)^3} e^{-\frac{1}{2\sigma^2} \left( \frac{\frac{\eta}{e^x} - \frac{\eta}{1-e^x}}{\frac{\eta}{1-e^x}} \right)^2}$$

$$\left[ 1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2} \left( \frac{\frac{\eta}{e^x} - \frac{\eta}{1-e^x}}{\frac{\eta}{1-e^x}} \right)^2} \right]$$

(40)

Figures 1 and 2 show the behaviour of the CDF and PDF of the OTRIED for a few varying values of the parameters and furthermore, for a simplicity,  $\sigma = a$ ,  $\lambda = b$  and  $\eta = d$ .

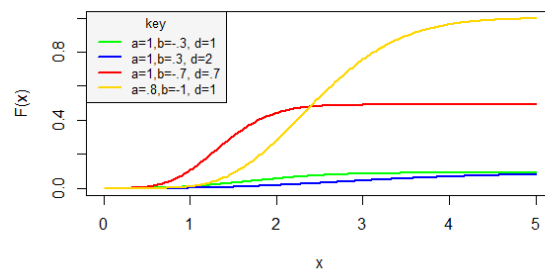


Figure 1: The CDF plot of OTRIED

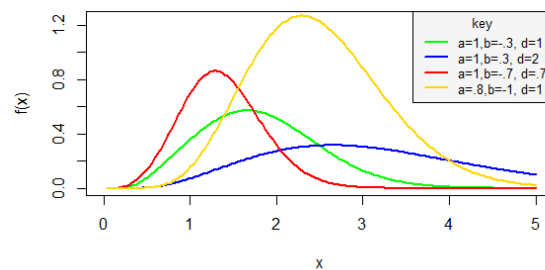


Figure 2: The PDF plot of OTRIED

### 6.1. Survival function of OTRIED

The survival function of OTRIED is defined as:

$$S(x; \sigma, \lambda, \eta) = 1 - F(x; \sigma, \lambda, \eta)$$

$$S(x; \sigma, \lambda, \eta) = 1 - \left( 1 - e^{-\frac{1}{2\sigma^2} \left( \frac{\frac{\eta}{e^x} - \frac{\eta}{1-e^x}}{\frac{\eta}{1-e^x}} \right)^2} \right) \left( 1 + \lambda e^{-\frac{1}{2\sigma^2} \left( \frac{\frac{\eta}{e^x} - \frac{\eta}{1-e^x}}{\frac{\eta}{1-e^x}} \right)^2} \right)$$

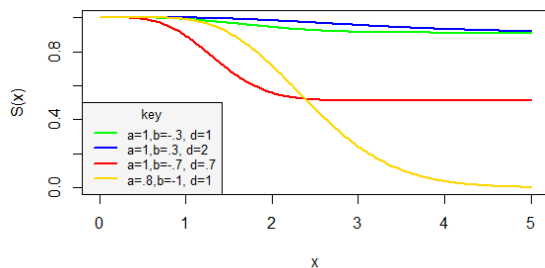
(41)

**6.2. Hazard function of OTRIED**

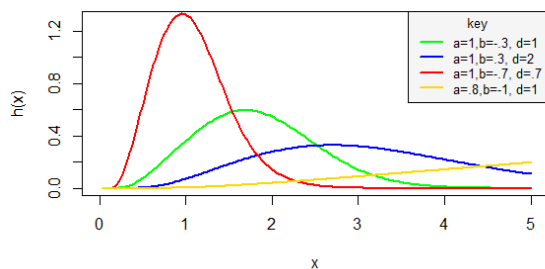
The hazard function of OTRIED is defined as:

$$H(x; \sigma, \lambda, \eta) = \frac{f(x; \sigma, \lambda, \eta)}{1 - F(x; \sigma, \lambda, \eta)} = \frac{\eta e^{-\frac{2\eta}{x}} e^{-\frac{1}{2\sigma^2} \left(\frac{\eta}{e^x - 1}\right)^{-2}} \left[ 1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2} \left(\frac{\eta}{e^x - 1}\right)^{-2}} \right]}{x^2 \sigma^2 \left( 1 - e^{-\frac{\eta}{x}} \right)^3 \left( 1 - \left( 1 - e^{-\frac{1}{2\sigma^2} \left(\frac{\eta}{e^x - 1}\right)^{-2}} \right) \left( 1 + \lambda e^{-\frac{1}{2\sigma^2} \left(\frac{\eta}{e^x - 1}\right)^{-2}} \right) \right)} \tag{42}$$

Below are the presentations of the behaviour of the survival and hazard functions of the OTRIED for a few varying parameters' values.



**Figure 3: The Survival function plot of OTRIED**



**Figure 4: The Hazard function plot of OTRIED**

**6.3. Estimation of Parameters of the OTRIED**

Here, the new generated distribution's parameters are computed using the MLE technique. With sample values  $x_1, x_2, \dots, x_n$  and joint density

functions as  $f(x_1, x_2, \dots, x_n; \Theta)$ , where  $\Theta = (\sigma, \lambda, \eta)^T$  is a vector of an unknown parameter, let  $X_1, X_2, \dots, X_n$  be  $n$  RV, from a population that follows OTRIED. The  $\ell$  is therefore described as follows:

$$\ell = -2n \log \sigma + \sum_{i=1}^n \log g(x_i; \phi) + \sum_{i=1}^n \log G(x_i; \phi) - 3 \sum_{i=1}^n \log \bar{G}(x_i; \phi) - \frac{1}{2\sigma^2} \sum_{i=1}^n \left( e^x - 1 \right)^{-2} + \sum_{i=1}^n \log \left( 1 - \lambda + 2\lambda e^{-\frac{1}{2\sigma^2} \left(\frac{\eta}{e^x - 1}\right)^{-2}} \right) \tag{43}$$

By performing partially differentiation for  $\ell$  with regard to  $\lambda, \sigma$  and  $\eta$  and equating the result to 0, one can derive the component of the score vector,  $U(\Theta)$ .

**7.0 APPLICATIONS**

In this study, the new distribution (OTRIED) was practicalized on two different real-life data to assess its flexibility. The MLEs of the model's parameters are computed using R software. The performance of the generated distribution (OTRIED), which belongs to the OTR-X, was compared to some related distributions from the literature using two distinct real datasets. The distributions are TIED, EIED, and IED. Furthermore, the proposed distribution can be practicalized to any skewed dataset.

**7.1. The first Real Dataset**

The dataset, which reflects the number of years a group of patients who got only chemotherapy survived, is a part of the data described by some scholars, named Bekker et al. (2000). This dataset contains the times of survival expressed in years for some patients. The details of the data are: 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 3.978, 4.003, 4.033.

**7.2. The second Real dataset**

The second data used in this research was data on the remission times of a random sample of some patients with bladder cancer. Oguntunde et al. (2016) also used the data.

0.08, 2.09, 3.48,4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57,5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54,3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64,3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05,2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26,2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02,4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25,8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53,

12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07,3.36, 6.93, 8.65, 12.63, 22.69.

**7.3. Statistics used for Comparing the Distributions**

The  $\ell$ , Akaike Information (AI) Criterion (AIC), Consistent AI Criterion (CAIC), Bayesian Information Criterion (BIC), and Hannan-Quinn Information Criteria (HQIC) were some of the metrics we took into consideration when comparing the proposed distributions. These statistics are expressed mathematically as;

$$AIC = -2\ell + 2p, \quad BIC = -2\ell + p \log(n),$$

$$CAIC = -2\ell + \frac{2pn}{(n-p-1)} \quad \text{and}$$

$$HQIC = -2\ell + 2p \log(\log(n))$$

where,  $n$  stands for size,  $\ell$  stands as log-likelihood function,  $p$  illustrate the quantity of distribution's parameters.

**Table 1: Estimates of the OTRIED and the competing models' parameters**

Model	$\sigma$	$\lambda$	$\hat{\eta}$	$\alpha$
OTRIED	0.4434	0.5000	0.6200	-
TIED	-	0.7531	0.8474	-
EIED	-	-	0.8965	0.9214
IED	-	-	0.8853	-

**Table 2: Results of the Goodness-of-Fit Statistics for the Dataset**

Model	$-\ell$	AIC	CAIC	HQIC	BIC	RANK
OTRIED	-19.6049	-33.2097	-32.6243	-31.1892	-27.7897	1
TIED	76.2402	156.4803	156.7660	157.8273	160.0936	2
EIED	79.8567	163.7134	163.9991	165.0604	167.3267	3
IED	82.2165	166.4330	166.5261	167.1065	168.2397	4

**Table 3: Estimates of the OTRIED and the competing models' parameters**

Model	$\sigma$	$\lambda$	$\hat{\eta}$	$\alpha$
OTRIED	0.2046	0.5000	5.4071	-
TIED	-	0.0016	2.5305	-
EIED	-	-	0.3941	0.0499
IED	-	-	2.4847	-

**Table 4: Results of the Goodness-of-Fit Statistics for the Dataset**

Model	$-\ell$	AIC	CAIC	HQIC	BIC	RANK
OTRIED	99.4522	204.9044	205.0980	208.3808	213.4605	1
TIED	406.4460	924.8938	924.9898	927.2114	930.5959	4

EIED	415.5845	835.1691	835.2651	837.4867	840.8732	2
IED	406.3823	922.7646	922.7963	923.9234	925.6166	3

## 8.0 DISCUSSION

Tables 1 and 3 shows the MLEs of the parameters of the OTRIED and that of the competing distributions obtained from existing literature while, in tables 2 and 4 we can deduce that the proposed distribution possessed the least values of the goodness of fit statistics and therefore is considered as the best model for the first and the second dataset respectively in this study.

## 9.0 CONCLUSION

Odd Transmuted Rayleigh-X is a brand-new class of probability distributions with two parameters. Consequently, the key statistical features of the OTR-X class of distributions have been derived and adequately presented. Some definitions for the distribution function, PDF, survival and hazard function, ordered statistics, and

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asymptotic behaviour were shown. Thus, both CDF and PDF of the OTR-X were likewise expanded to ease computation of some features of the OTR-X family as they belonged to the Exponentiated- G class. The MLE approach was adopted in estimating the family's parameters. In comparison to TIED, EIED, and IED, a new sub-model (OTRIED) that was generated from the OTR-X class has proven to be a better fit when fitted to two real-life datasets.

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