



The One-Dimensional Non-Homogeneous Wave Equation

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Abstract

The well-known ideal wave equation is valid only for homogeneous media, where the wave propagation speed must be constant. In this paper, a modification of the wave equation was made so that it describes a disturbance propagating in a non-homogeneous medium. Two examples of this equation were worked out.

Keywords: *Wave equation; homogeneous media; propagation speed; disturbance; non-homogeneous medium.*

1.0 INTRODUCTION

Wave equation is an important second-order partial differential equation for the description of waves as they occur in physics such as sound waves, light waves and water waves. It arises in fields like acoustics, electromagnetics and fluid dynamics (Lipson, 1969; Whitham, 1974).

The one-dimensional wave equation is given by:

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \quad (1)$$

where v is a constant, the velocity of the wave (Courant and Hilbert, 1962). Solutions to equation (1) describes any disturbance (wave) propagating in a homogeneous medium. However, if the

medium is not homogenous, equation (1) cannot be used to describe the motion of the wave (Goldstein, 1980; John, 1986; Zhu; 1986). The non-homogeneity is manifested in the velocity of the disturbance not being constant, but depending position. Many authors did not heed this point. See references for example (Lipson, 1969; Treves, 1975; Goldstein, 1980; Zhu, 1986; Kevorkian, 1989).

1.1 Objectives

The objective of this work is to modify wave equation such that it describes a disturbance that propagates in a non-homogeneous medium.

2.0 DERIVATION

If $\Psi = \Psi(x, t)$ represents a one-dimensional disturbance or wave, we can write:

$$\frac{\partial^2 \psi}{\partial t} = \frac{\partial \psi}{\partial x} \frac{\partial x}{\partial t} = \frac{\partial \psi}{\partial x} v \quad (2)$$

where v is the velocity of the disturbance. Differentiating equation (2) with respect to the spatial coordinate x gives:

$$\frac{\partial^2 \psi}{\partial x \partial t} = \frac{\partial^2 \psi}{\partial x^2} v + \frac{\partial \psi}{\partial x} \frac{\partial v}{\partial x} \quad (3)$$

If the velocity v is assumed to be independent of time, differentiating equation (2) with respect to time gives:

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x \partial t} v \quad (4)$$

Equating the mixed second derivative terms in equations (3) and (4) produced:

$$\frac{\partial^2 \psi}{\partial x^2} v + \frac{\partial \psi}{\partial x} \frac{\partial v}{\partial x} = \frac{1}{v} \frac{\partial^2 \psi}{\partial t^2} \quad (5)$$

Rearranging the terms in equation (5) yields:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{v} \frac{\partial \psi}{\partial x} \frac{\partial v}{\partial x} - \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \quad (6)$$

Equation (6) is the non-homogeneous wave equation in one dimension. Notice that when v is a constant, the middle term in equation (6) vanishes and we obtain the ordinary wave equation (equation (1)) as a special case.

3.0 SOLUTIONS

It can be shown by direct substitution (see appendix) that the general solution to equation (6) has the form:

$$\Psi(x, t) = G\left(x - \int_0^t v(t') dt'\right) + F\left(x + \int_0^t v(t') dt'\right), \quad (7)$$

where $v(t)$ is the velocity of the wave front as a function of time, and F and G are continuous functions having as their argument the term $x \pm \int v(t') dt'$

One can also think of a solution to equation (6) that is of the form:

$$\Psi(x, t) = X(x)T(t), \quad (8)$$

where $X(x)$ is only a function of position, x , and $T(t)$ is only a function of time, t . Substituting equation (8) into equation (6) and separating the variables produces two ordinary differential equations:

$$\frac{d^2 T}{dt^2} + w^2 T = 0, \quad (9)$$

and

$$\frac{d^2 X}{dx^2} + \frac{1}{v} \frac{\partial v}{\partial x} \frac{dX}{dx} + \frac{w^2}{v^2} X = 0, \quad (10)$$

Where w^2 is the separation constant. Equation (9) is the equation that describes the motion of a simple harmonic oscillator. Equation (10) is the one-dimensional time-independent wave equation for a non-homogeneous medium.

Equation (9) has a significant physical implication: the frequency of the wave in a non-homogeneous medium is constant, the effect of the inhomogeneity is reflected only upon the wavelength and the velocity.

4.0 EXAMPLES

4.1 A Medium with Varying Index of Refraction

In an optically non-homogeneous medium, the refractive index is not constant throughout the medium, but could depend upon many variables (Emil, 1999). For simplicity, we will assume a one-dimensional dependence of the refractive index:

$$v(x) = \frac{c}{n(x)} \quad \text{and} \quad \frac{dv}{dx} = -\frac{c}{n^2(x)} \frac{dn}{dx}, \quad (11)$$

Substituting equations (11) into equation (10) to get:

$$\frac{d^2 X}{dx^2} - \frac{1}{n(x)} \frac{dn}{dx} \frac{dX}{dx} + \left[\frac{w n(x)}{c}\right]^2 X = 0, \quad (12)$$

If we know how the index of refraction depends upon position, we can solve this linear differential equation for the space function $X(x)$.

The two independent solutions of equation (12) take on the form:

$$X(x) = A \exp \left\{ i \frac{w}{c} \int_0^x n(x') dx' \right\} + B \exp \left\{ -i \frac{w}{c} \int_0^x n(x') dx' \right\} \quad (13)$$

where A and B are arbitrary constants.

4.2 Mechanical (Sound) Waves in a Medium of Varying Density

In a medium of varying density, the velocity of a propagating sound wave is related to the density of the medium as (Everest, 2001):

$$v^2(x) = \frac{B}{\rho(x)} \quad , \quad (14)$$

where B is the bulk modulus of the medium and $\rho(x)$ is the density of the medium. We will assume that the bulk modulus is constant. For this situation, equations (11) become:

$$v(x) = \sqrt{\frac{B}{\rho(x)}} \quad \text{and} \quad \frac{dy}{dx} = -\frac{1}{2} \sqrt{\frac{B}{\rho^3(x)}} \frac{d\rho(x)}{dx} \quad , \quad (15)$$

Substituting equations (15) into equation (10) gives:

$$\frac{d^2X}{dx^2} + \frac{1}{2} \frac{1}{\rho(x)} \frac{d\rho(x)}{dx} \frac{dX}{dx} + \frac{\rho(x) w^2}{B} X = 0 \quad , \quad (16)$$

In the same manner as before, the general solution to equation (16) has the form:

$$X(x) = A \exp \left\{ i \frac{w}{\sqrt{B}} \int_0^x \sqrt{\rho(x')} dx' \right\} + B \exp \left\{ -i \frac{w}{\sqrt{B}} \int_0^x \sqrt{\rho(x')} dx' \right\} \quad , \quad (17)$$

where A and B are arbitrary constants.

5.0 CONCLUSION

We conclude that, in quantum theory, equation (6) may be a modified version of the Schrödinger equation, with v , the velocity, representing the group velocity and Ψ representing the motion of the expectation value.

6.0 REFERENCES

- Courant, R, Hilbert, D. (1962). *Methods of Mathematical Physics, vol II*. Interscience (Wiley) New York.
- Emil, M.W (1999). Principles of Optics, 7th expanded Edition, p. 376. ISBN 978-0-521-78449-8.
- Everest, F. (2001). The Master Handbook of Acoustics. New York: McGraw-Hill. P.262-263. ISBN 0- 07-1366097-2.
- Goldstein, H. (1980). Classical_Mechanics, Addison Wesley, 2nd Edition, p. 487-489.
- John, F. (1986). Partial Differential Equations, 4th Edition, Springer p.536.
- Kevorkian, J. (1989). Partial Differential Equations -Analytical Solution Techniques, p.547. ISBN 0-534-12216-7 (wadsworth and Brooks/Cole).
- Lipson, S. (1969). Optical Physics, Cambridge University Press, UK, p. 89.
- Treves, F. (1975). Basic Linear Partial Differential Equations, 3rd Edition, Academic Press.p.34
- Whitham, G. B. (1974). Linear and non-linear waves, John Wiley and Sons Inc, N.Y. p. 17-67. ISBN 978-0-471-35942-5
- Zhu, S. (1986). Frustrated Total Internal Reflection: A demonstration and review, American Journal of Physics, 54, 7th July, p. 604.

APPENDIX

Proof by direct substitution begins with equation (7)

$$\Psi(x, t) = G \left(x - \int_0^t v(t') dt' \right) + F \left(x + \int_0^t v(t') dt' \right) \quad ,$$

The second derivative with respect to position is:

$$\frac{\partial^2}{\partial x^2} \Psi(x, t) = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 F}{\partial x^2} \quad ,$$

The first derivative with respect to time is:

$$\frac{\partial}{\partial t} \Psi(x, t) = \frac{\partial G}{\partial u_-} (-v) + \frac{\partial F}{\partial u_+} (v) \quad ,$$

where $u_{\pm} = x \pm \int_0^t v(t') dt'$

Taking the second derivative gives:

$$\frac{\partial^2}{\partial t^2} \Psi(x, t) = \frac{\partial^2 G}{\partial u_{\pm}^2} (-v)^2 + \frac{\partial G}{\partial u_{\pm}} \frac{\partial v}{\partial t} + \frac{\partial^2 F}{\partial u_{\pm}^2} (v)^2 + \frac{\partial F}{\partial u_{\pm}} \frac{\partial v}{\partial t} ,$$

Using the fact that $\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial t} = \frac{\partial v}{\partial x} v$,

Equation (6) becomes:

$$\left(\frac{\partial^2 G}{\partial x^2} - \frac{\partial^2 G}{\partial u_{\pm}^2} \right) + \left(\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial u_{\pm}^2} \right) + \frac{1}{v} \frac{\partial v}{\partial x} \left[\left(\frac{\partial G}{\partial x} - \frac{\partial G}{\partial u_{\pm}} \right) + \left(\frac{\partial F}{\partial x} - \frac{\partial F}{\partial u_{\pm}} \right) \right] ,$$

The terms in the round brackets vanish because:

$$\frac{\partial}{\partial \pm} = \frac{\partial x}{\partial u_{\pm}} \frac{\partial}{\partial x} = \frac{1}{\frac{\partial u_{\pm}}{\partial x}} \frac{\partial}{\partial x} = \frac{\partial}{\partial x} ,$$

thus showing that equation (7) is a solution to equation (6) by direct substitution.
