



Continuous Implicit One Step Four-Point Hybrid Block Method for The Solution of Initial Value Problem for Third Order Ordinary Differential Equations

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ABSTRACT

This paper focuses on the development of a continuous hybrid method for direct solution of third order initial value problems (IVPs) of ordinary differential equations (ODEs) using power series as the basis function. The approximate solution was interpolated at all grid and off-grid points except at the point of evaluation and the third derivative of the approximate solution was collocated at the grid points to generate a system of linear equations for the determination of the unknown parameters. The derived method was tested for consistency, zero stability, convergence and absolute stability. The method competes effectively when results are compared with those existing in literatures.

Keywords: hybrid method, interpolation, collocation, power series

1.0 INTRODUCTION

Considered the third order initial value problem of the form

$$\begin{aligned} y''' &= f(x, y, y', y''), \\ y(\alpha) &= \mu_0, y'(\alpha) = \mu_1, y''(\alpha) = \mu_2 \end{aligned} \quad (1)$$

Before now, researchers in most cases always use method of reduction of higher order ODEs into system of first order ODEs to solve (1). This technique though quite good, but has many problems such as very tediousness, complexity of the method, time consuming and the need for large computer storage memories because of too many auxiliary functions, etc. This approach has been extensively discussed in Awoyemi (2001), Awoyemi (2003) and Awoyemi and Kayode (2005). Direct methods for higher order ODEs are developed to avoid the limitation of the methods mentioned above, Adesanya, Udoh and Alkali (2012), adopted predictor-corrector approach, which has also been reported to be inefficient

because the cost of developing separate predictors, human and computer time involved in the execution of the method are too costly. Recently, in order to remove the difficulties usually encountered by adopting this mode of solution, researchers, Jator and Li (2009), James, Adesanya and Joshua (2013), Adesanya, Udoh and Ajileye (2013), Awoyemi and Idowu (2005), Kayode and Obarhua (2017), Kuboye and Omar (2015), Areo and Adeniyi (2014) and Olabode (2013).

Areo and Adeniyi (2014) and Olabode (2013) have proposed direct methods which is better than Predictor-Corrector methods implemented in block-by-block manner. The desire for numerical methods with better accuracy has also led to the introduction of hybrid linear multistep methods which have recorded high success since its introduction.

In this paper, we considered continuous implicit one-step 4-point hybrid block method for the solution of initial value problem of general third order ordinary differential equation. Followed by analysis of the proposed method to establish the numerical stability, numerical example to demonstrate the efficiency advantages of the proposed method and subsequently.

2.0 DERIVATION OF THE METHOD

We consider a power series approximation of the form

$$p(x) = \sum_{j=0}^{u+v-1} a_j x^j \quad (2)$$

where u and v are the interpolation and collocation points respectively, and $h = x_{n-1} - x_n$, $n = 0, 1, \dots, N$ is the constant step size of the partition in the interval $[a, b]$. Differentiating equation (2) thrice, gives

$$p'''(x) = \left. \sum_{j=3}^{u+v-1} j(j-1)(j-2)a_j x^{j-3} = f(x, y, y', y'') \right\} \quad (3)$$

Interpolating equation (2) at x_{n+u} , $u = \left\{ 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \right\}$ and collocating (3)

at points x_{n+v} , $v = \{0, 1\}$, leads to the following system of non-linear equations of the form $AX = U$ (4)

where

$$X = \left[\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_0, \beta_1 \right]^T$$

$$U = \left[y_n, y_{n+\frac{1}{5}}, y_{n+\frac{2}{5}}, y_{n+\frac{3}{5}}, y_{n+\frac{4}{5}}, f_n, f_{n+1} \right]^T$$

A=

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_n + \frac{1}{5}h & \left(x_n + \frac{1}{5}h\right)^2 & \left(x_n + \frac{1}{5}h\right)^3 & \left(x_n + \frac{1}{5}h\right)^4 & \left(x_n + \frac{1}{5}h\right)^5 & \left(x_n + \frac{1}{5}h\right)^6 \\ 1 & x_n + \frac{2}{5}h & \left(x_n + \frac{2}{5}h\right)^2 & \left(x_n + \frac{2}{5}h\right)^3 & \left(x_n + \frac{2}{5}h\right)^4 & \left(x_n + \frac{2}{5}h\right)^5 & \left(x_n + \frac{2}{5}h\right)^6 \\ 1 & x_n + \frac{3}{5}h & \left(x_n + \frac{3}{5}h\right)^2 & \left(x_n + \frac{3}{5}h\right)^3 & \left(x_n + \frac{3}{5}h\right)^4 & \left(x_n + \frac{3}{5}h\right)^5 & \left(x_n + \frac{3}{5}h\right)^6 \\ 1 & x_n + \frac{4}{5}h & \left(x_n + \frac{4}{5}h\right)^2 & \left(x_n + \frac{4}{5}h\right)^3 & \left(x_n + \frac{4}{5}h\right)^4 & \left(x_n + \frac{4}{5}h\right)^5 & \left(x_n + \frac{4}{5}h\right)^6 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 \\ 0 & 0 & 0 & 6 & 24x_n + 24h & 60(x_n + h)^2 & 120(x_n + h)^3 \end{bmatrix}$$

Solving (4) for the unknown constants α 's and β 's using Gaussian elimination and substituting it back into (1) and after some algebraic manipulations yield, a continuous linear multistep method of the form

$$p(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+\frac{1}{5}} + \alpha_2(x)y_{n+\frac{2}{5}} + \alpha_3(x)y_{n+\frac{3}{5}} + \alpha_4(x)y_{n+\frac{4}{5}} + h^2 \sum_{j=0}^1 \beta_j(x)f_{n+j} \quad (5)$$

Setting $k = 1$ as the step-size, the coefficient of α 's and β 's are expressed as function of

$t = \frac{x - x_n}{h}$ as follows

$$\alpha_0 = 1 - \frac{335}{36}t + \frac{5135}{216}t^2 - \frac{15625}{216}t^4 + \frac{3125}{36}t^5 - \frac{3125}{108}t^6$$

$$\alpha_1 = \frac{434}{27}t - \frac{10235}{162}t^2 + \frac{165625}{684}t^4 - \frac{8750}{27}t^5 + \frac{71875}{648}t^6$$

$$\alpha_2 = -\frac{89}{9}t + \frac{6415}{108}t^2 - \frac{71875}{216}t^4 + \frac{8125}{18}t^5 - \frac{34375}{316}t^6$$

$$\alpha_3 = \frac{34}{9}t - \frac{1315}{54}t^2 + \frac{40625}{216}t^4 - \frac{2500}{9}t^5 + \frac{21875}{216}t^6$$

$$\alpha_4 = -\frac{71}{108}t + \frac{2825}{648}t^2 - \frac{3125}{81}t^4 + \frac{6875}{108}t^5 - \frac{15625}{648}t^6$$

$$\beta_0 = \frac{13}{3375}ht - \frac{1727}{40500}t^2 + \frac{1}{6}t^3 - \frac{379}{1296}t^4 + \frac{25}{108}t^5 - \frac{85}{1296}t^6$$

$$\beta_1 = -\frac{1}{3375}ht - \frac{83}{40500}t^2 + \frac{29}{1296}t^4 - \frac{5}{108}t^5 + \frac{35}{1296}t^6 \quad (6)$$

Evaluating (6) at $t = 1$ gives the discrete scheme

$$y_{n+1} = y_n - \frac{14}{3}y_{n+\frac{1}{5}} + 9y_{n+\frac{2}{5}} - 9y_{n+\frac{3}{5}} + \frac{14}{3}y_{n+\frac{4}{5}} + \frac{1}{750}h^3f_n + \frac{1}{750}h^3f_{n+1} \quad (7)$$

To obtain the hybrid block method, additional equations is required by evaluating the first and second derivative of (6) at

$$x = x_n, x_{n+\frac{1}{5}}, x_{n+\frac{2}{5}}, x_{n+\frac{3}{5}}, x_{n+\frac{4}{5}}, \quad \text{while the remaining equation is obtained by evaluating third derivative of (6) at } x = x_n, x_{n+\frac{1}{5}} \text{ and } x_{n+\frac{2}{5}} \text{ to give the following equations}$$

$$hy'_n = \frac{335}{36}y_n + \frac{434}{27}y_{n+\frac{1}{5}} - \frac{89}{9}y_{n+\frac{2}{5}} + \frac{34}{9}y_{n+\frac{3}{5}} - \frac{71}{108}y_{n+\frac{4}{5}} + \frac{13}{3375}h^3f_n + \frac{1}{3375}h^3f_{n+1}$$

$$hy'_{n+\frac{1}{5}} = -\frac{53}{36}y_n - \frac{367}{108}y_{n+\frac{1}{5}} + \frac{235}{36}y_{n+\frac{2}{5}} - \frac{71}{36}y_{n+\frac{3}{5}} + \frac{17}{54}y_{n+\frac{4}{5}} - \frac{113}{135000}h^3f_n - \frac{17}{135000}h^3f_{n+1}$$

$$hy'_{n+\frac{2}{5}} = \frac{19}{36}y_n - \frac{100}{27}y_{n+\frac{1}{5}} + \frac{4}{9}y_{n+\frac{2}{5}} + \frac{28}{9}y_{n+\frac{3}{5}} - \frac{41}{108}y_{n+\frac{4}{5}} + \frac{8}{16875}h^3f_n + \frac{2}{16875}h^3f_{n+1}$$

$$hy'_{n+\frac{3}{5}} = -\frac{19}{36}y_n + \frac{307}{108}y_{n+\frac{1}{5}} - \frac{283}{36}y_{n+\frac{2}{5}} + \frac{155}{36}y_{n+\frac{3}{5}} + \frac{67}{54}y_{n+\frac{4}{5}} - \frac{79}{135000}h^3f_n - \frac{31}{135000}h^3f_{n+1}$$

$$hy'_{n+\frac{4}{5}} = \frac{53}{36}y_n - \frac{194}{27}y_{n+\frac{1}{5}} + \frac{137}{9}y_{n+\frac{2}{5}} - \frac{178}{9}y_{n+\frac{3}{5}} + \frac{1109}{108}y_{n+\frac{4}{5}} + \frac{31}{16875}h^3f_n + \frac{19}{16875}h^3f_{n+1}$$

$$hy'_{n+1} = \frac{335}{36}y_n - \frac{4619}{108}y_{n+\frac{1}{5}} + \frac{2879}{36}y_{n+\frac{2}{5}} - \frac{2659}{36}y_{n+\frac{3}{5}} + \frac{1477}{54}y_{n+\frac{4}{5}} + \frac{343}{27000}h^3f_n + \frac{439}{27000}h^3f_{n+1}$$

$$h^2y''_n = \frac{5135}{108}y_n - \frac{10235}{81}y_{n+\frac{1}{5}} + \frac{6415}{54}y_{n+\frac{2}{5}} - \frac{1315}{27}y_{n+\frac{3}{5}} + \frac{2825}{324}y_{n+\frac{4}{5}} - \frac{1727}{20250}h^3f_n - \frac{83}{20250}h^3f_{n+1}$$

$$h^2y''_{n+\frac{1}{5}} = \frac{2735}{108}y_n - \frac{16265}{324}y_{n+\frac{1}{5}} + \frac{2555}{108}y_{n+\frac{2}{5}} + \frac{215}{108}y_{n+\frac{3}{5}} - \frac{125}{162}y_{n+\frac{4}{5}} + \frac{667}{81000}h^3f_n + \frac{43}{81000}h^3f_{n+1}$$

$$h^2y''_{n+\frac{2}{5}} = -\frac{265}{108}y_n + \frac{2815}{81}y_{n+\frac{1}{5}} - \frac{3485}{54}y_{n+\frac{2}{5}} + \frac{935}{27}y_{n+\frac{3}{5}} - \frac{775}{324}y_{n+\frac{4}{5}} - \frac{17}{20250}h^3f_n + \frac{7}{20250}h^3f_{n+1}$$

$$h^2y''_{n+\frac{3}{5}} = -\frac{265}{108}y_n + \frac{2935}{324}y_{n+\frac{1}{5}} + \frac{1355}{108}y_{n+\frac{2}{5}} - \frac{4585}{108}y_{n+\frac{3}{5}} + \frac{3775}{162}y_{n+\frac{4}{5}} - \frac{293}{81000}h^3f_n - \frac{197}{81000}h^3f_{n+1}$$

$$h^2y''_{n+\frac{4}{5}} = \frac{2735}{108}y_n - \frac{9635}{81}y_{n+\frac{1}{5}} + \frac{12415}{54}y_{n+\frac{2}{5}} - \frac{5515}{27}y_{n+\frac{3}{5}} + \frac{22025}{324}y_{n+\frac{4}{5}} + \frac{673}{20250}h^3f_n + \frac{517}{20250}h^3f_{n+1}$$

$$h^2y''_{n+1} = \frac{5135}{108}y_n - \frac{69065}{324}y_{n+\frac{1}{5}} + \frac{40955}{108}y_{n+\frac{2}{5}} - \frac{33385}{108}y_{n+\frac{3}{5}} + \frac{15475}{162}y_{n+\frac{4}{5}} + \frac{5467}{81000}h^3f_n + \frac{12043}{81000}h^3f_{n+1}$$

$$h^3f'_{n+\frac{1}{5}} = -\frac{500}{3}y_n + \frac{5000}{9}y_{n+\frac{1}{5}} - \frac{2000}{3}y_{n+\frac{2}{5}} + \frac{1000}{3}y_{n+\frac{3}{5}} - \frac{500}{9}y_{n+\frac{4}{5}} + \frac{4}{45}h^3f_n + \frac{1}{45}h^3f_{n+1}$$

$$h^3f'_{n+\frac{2}{5}} = -\frac{250}{3}y_n + \frac{1750}{9}y_{n+\frac{1}{5}} - \frac{250}{3}y_{n+\frac{2}{5}} - \frac{250}{3}y_{n+\frac{3}{5}} + \frac{500}{9}y_{n+\frac{4}{5}} - \frac{4}{45}h^3f_n - \frac{1}{45}h^3f_{n+1}$$

(8)

2.1 Development of the Block Method

we defined a modified block method on (8) as

$$h^\eta \sum_{j=0}^q \alpha_j(x) y_{n+j} =$$

$$h^\eta \sum_{j=0}^q e_{ij} y_n^\eta + h^{\lambda-\eta} \left[\sum_{j=1}^q d_{ij} f_n + \sum_{j=1}^q b_{ij} f_{n+j} \right] \quad (9)$$

$$i = 0, 1, \dots, q$$

Where η is the power of the derivative of the continuous method and λ is the order of the problem to solved; $q = u + v$

Writing (9) in vector notation gives

$$h^\eta \bar{a} Y_m = h^\eta \bar{e} y_m + h^{\lambda-\eta} [\bar{d} f(y_m) + \bar{b} F(Y_m)] \quad (10)$$

The matrices $\bar{a} = (a_0)$, $\bar{b} = (b_0)$, $\bar{e} = (e_0)$, $\bar{d} = (d_0)$ are constant coefficients matrices and our block method is in the form

$$IY_L = \bar{B}R_1 + \bar{C}R_2 + \bar{D}R_3 \quad (11)$$

solving equation (7) and (8) together using matrix inversion to obtain the coefficients of (11) of the new block method as

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Y_L = \begin{bmatrix} y_{n+\frac{1}{5}}, y_{n+\frac{2}{5}}, y_{n+\frac{3}{5}}, y_{n+\frac{4}{5}}, y_{n+1}, y'_{n+\frac{1}{5}}, y'_{n+\frac{2}{5}}, \\ y'_{n+\frac{3}{5}}, y'_{n+\frac{4}{5}}, y'_{n+1}, y''_{n+\frac{1}{5}}, y''_{n+\frac{2}{5}}, y''_{n+\frac{3}{5}}, y''_{n+\frac{4}{5}}, y''_{n+1} \end{bmatrix}^T$$

$$R_1 = [y_n, y'_n, y''_n]^T$$

$$R_2 = [f_n]$$

$$R_3 = \left[f_{n+\frac{1}{5}}, f_{n+\frac{2}{5}}, f_{n+1} \right]^T$$

$$\bar{D} = \begin{bmatrix} \frac{37h^3}{6000} & \frac{7h^3}{45000} & \frac{h^3}{180000} \\ \frac{13h^3}{1875} & \frac{h^3}{1125} & \frac{h^3}{28125} \\ \frac{459h^3}{20000} & \frac{9h^3}{5000} & \frac{9h^3}{100000} \\ \frac{16h^3}{375} & \frac{112h^3}{5625} & \frac{16h^3}{28125} \\ \frac{5h^3}{96} & \frac{5h^3}{72} & \frac{h^3}{288} \\ \frac{17h^2}{1500} & \frac{23h^2}{9000} & \frac{h^2}{11250} \\ \frac{7h^2}{125} & \frac{2h^2}{1125} & \frac{h^2}{5625} \\ \frac{99h^2}{1000} & \frac{39h^2}{1000} & \frac{3h^2}{5000} \\ \frac{32h^2}{325} & \frac{176h^2}{1125} & \frac{32h^2}{5625} \\ 0 & \frac{25h^2}{72} & \frac{h^2}{36} \\ \frac{7h}{48} & \frac{h}{40} & \frac{h}{1200} \\ \frac{4h}{15} & \frac{h}{15} & 0 \\ \frac{9h}{80} & \frac{3h}{8} & \frac{3h}{400} \\ \frac{4h}{15} & \frac{4h}{5} & \frac{4h}{75} \\ \frac{h}{15} & \frac{25h}{24} & \frac{6h}{16} \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} 1 & \frac{h}{5} & \frac{h^2}{50} \\ 1 & \frac{2h}{5} & \frac{2h^2}{25} \\ 1 & \frac{3h}{5} & \frac{9h^2}{50} \\ 1 & \frac{4h}{5} & \frac{8h^2}{25} \\ 1 & h & \frac{h^2}{2} \\ 0 & 1 & \frac{h}{5} \\ 0 & 1 & \frac{2h}{5} \\ 0 & 1 & \frac{3h}{5} \\ 0 & 1 & \frac{4h}{5} \\ 0 & 1 & h \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} \frac{13h^3}{15000} \\ \frac{43h^3}{9375} \\ \frac{279h^3}{25000} \\ \frac{208h^3}{9375} \\ \frac{h^3}{24} \\ \frac{167h^2}{15000} \\ \frac{16h^2}{625} \\ \frac{207h^2}{5000} \\ \frac{136h^2}{1875} \\ \frac{h^2}{8} \\ \frac{47h}{600} \\ \frac{h}{15} \\ \frac{21h}{200} \\ \frac{16h}{75} \\ \frac{7h}{24} \end{bmatrix}$$

(11) can be express as

$$\left. \begin{aligned}
 y_{n+\frac{1}{5}} &= y_n + \frac{1}{5}hy'_n + \frac{1}{50}h^2y''_n + \frac{37}{60000}h^3f_{n+\frac{1}{5}} - \frac{7}{45000}h^3f_{n+\frac{2}{5}} + \frac{13}{15000}h^3f_n + \frac{1}{180000}h^3f_{n+1} \\
 y_{n+\frac{2}{5}} &= y_n + \frac{2}{5}hy'_n + \frac{2}{25}h^2y''_n + \frac{13}{1875}h^3f_{n+\frac{1}{5}} - \frac{1}{1125}h^3f_{n+\frac{2}{5}} + \frac{43}{9375}h^3f_n + \frac{1}{128125}h^3f_{n+1} \\
 y_{n+\frac{3}{5}} &= y_n + \frac{3}{5}hy'_n + \frac{9}{50}h^2y''_n + \frac{459}{20000}h^3f_{n+\frac{1}{5}} + \frac{9}{5000}h^3f_{n+\frac{2}{5}} + \frac{279}{25000}h^3f_n + \frac{9}{100000}h^3f_{n+1} \\
 y_{n+\frac{4}{5}} &= y_n + \frac{4}{5}hy'_n + \frac{8}{48}h^2y''_n + \frac{16}{375}h^3f_{n+\frac{1}{5}} + \frac{112}{5625}h^3f_{n+\frac{2}{5}} + \frac{208}{9375}h^3f_n + \frac{16}{28125}h^3f_{n+1} \\
 y_{n+1} &= y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{5}{96}h^3f_{n+\frac{1}{5}} + \frac{5}{72}h^3f_{n+\frac{2}{5}} + \frac{1}{24}h^3f_n + \frac{1}{288}h^3f_{n+1} \\
 y'_{n+\frac{1}{5}} &= y'_n + \frac{1}{5}hy''_n + \frac{17}{1500}h^2f_{n+\frac{1}{5}} - \frac{23}{9000}h^2f_{n+\frac{2}{5}} + \frac{167}{15000}h^2f_n + \frac{1}{11250}h^2f_{n+1} \\
 y'_{n+\frac{2}{5}} &= y'_n + \frac{2}{5}hy''_n + \frac{7}{125}h^2f_{n+\frac{1}{5}} - \frac{2}{1125}h^2f_{n+\frac{2}{5}} + \frac{16}{625}h^2f_n + \frac{1}{5625}h^2f_{n+1} \\
 y'_{n+\frac{3}{5}} &= y'_n + \frac{3}{5}hy''_n + \frac{99}{1000}h^2f_{n+\frac{1}{5}} + \frac{39}{1000}h^2f_{n+\frac{2}{5}} + \frac{207}{5000}h^2f_n + \frac{3}{5000}h^2f_{n+1} \\
 y'_{n+\frac{4}{5}} &= y'_n + \frac{4}{5}hy''_n + \frac{32}{375}h^2f_{n+\frac{1}{5}} + \frac{176}{1125}h^2f_{n+\frac{2}{5}} + \frac{136}{1875}h^2f_n + \frac{32}{5625}h^2f_{n+1} \\
 y'_{n+1} &= y'_n + hy''_n + \frac{25}{72}h^2f_{n+\frac{2}{5}} + \frac{1}{8}h^2f_n + \frac{1}{36}h^2f_{n+1} \\
 y''_{n+\frac{1}{5}} &= y''_n + \frac{7}{48}hf_{n+\frac{1}{5}} - \frac{1}{40}hf_{n+\frac{2}{5}} + \frac{47}{600}hf_n + \frac{1}{1200}h^3f_{n+1} \\
 y''_{n+\frac{2}{5}} &= y''_n + \frac{4}{15}hf_{n+\frac{1}{5}} + \frac{1}{15}hf_{n+\frac{2}{5}} + \frac{1}{15}hf_n \\
 y''_{n+\frac{3}{5}} &= y''_n + \frac{9}{48}hf_{n+\frac{1}{5}} + \frac{3}{8}hf_{n+\frac{2}{5}} + \frac{21}{200}hf_n + \frac{3}{400}hf_{n+1} \\
 y''_{n+\frac{4}{5}} &= y''_n - \frac{4}{15}hf_{n+\frac{1}{5}} + \frac{4}{5}hf_{n+\frac{2}{5}} + \frac{16}{75}hf_n + \frac{4}{75}hf_{n+1} \\
 y''_{n+1} &= y''_n - \frac{25}{48}hf_{n+\frac{1}{5}} + \frac{25}{24}hf_{n+\frac{2}{5}} + \frac{7}{24}hf_n + \frac{3}{16}hf_{n+1}
 \end{aligned} \right\} \quad (12)$$

2.2 Stability Properties of the Developed Method

2.2.1 Order and Error Constant

The four-point hybrid block method in (12) is of order six (6) with error constant

$$C_7 = \begin{bmatrix} \frac{41}{393750000}, & \frac{11}{16406250}, & \frac{9}{5468750}, \\ \frac{152}{24609375}, & \frac{1}{42000}, & \frac{37}{22500000}, \\ \frac{1}{281250}, & \frac{21}{2500000}, & \frac{32}{703125}, & \frac{1}{7200}, \\ \frac{17}{1125000}, & \frac{1}{281250}, & \frac{9}{125000}, & \frac{46}{140625}, & \frac{1}{1800} \end{bmatrix}^T$$

2.2.2 Consistency

The hybrid block method (12) is said to be consistent since it is of order six.

2.2.3 Zero Stability

Since

$$\left| z_i \right| = \left| 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1 \right| \leq 1$$

the method is zero-stable

2.2.4 Convergence

A block method is said to be convergent if and only if it is consistent and zero stable. From the above our method is convergent.

2.2.5 Region of Absolute Stability

To plot the region of absolute stability of the new method, the methods were formulated as general linear method and the stability polynomial for method is computed as

$$\begin{aligned} & \left(-\frac{1}{337500000} w^5 - \frac{83}{337500000} w^4 \right) h^9 \\ & + \left(-\frac{119}{37500000} w^5 - \frac{21259}{22500000} w^4 \right) h^6 \\ & + \left(-\frac{2}{625} w^5 - \frac{313}{1500} w^4 \right) + w^5 - \frac{5}{2} w^4 \end{aligned}$$

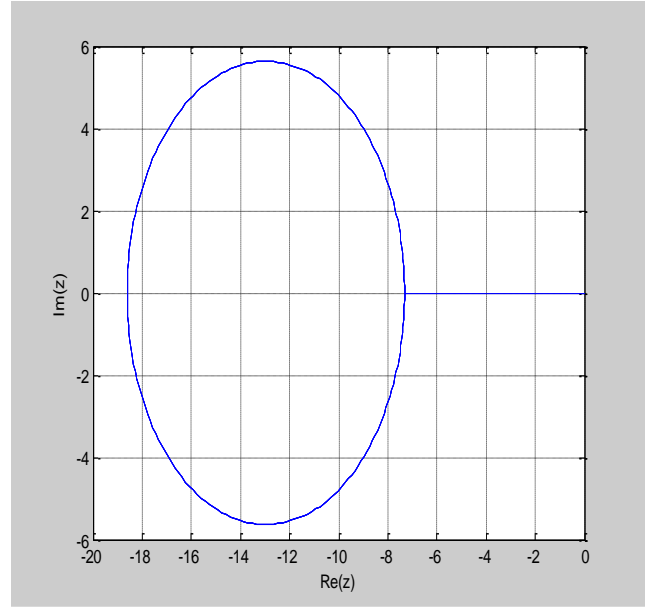


Figure 1. The Region of Absolute Stability of the developed method.

3.0 NUMERICAL EXPERIMENTS

We consider the following examples to test the efficiency of our method. The following notation are used in the tables below;

$y(x)$:- is the exact solution, y_n :- is the computed result at each case, $|error|$:- is the absolute error given by $(y_n - y(x))$.

Example 1

Stiff problem

$$y''' = 3 \sin x$$

$$y(0) = 1, y'(0) = 0, y''(0) = -2, h = \frac{1}{10}$$

Exact Solution:

$$y(x) = 3 \cos(x) + \frac{x^2}{2} - 2$$

Source: Olabode (2013).

Table 1: Comparison of new result with results from existing methods

| x | $y(x)$ | y_n | Error in New Result | Error in Olabode (2013) |
|-----|----------------------|------------------------|---------------------|-------------------------|
| 0.1 | 0.990012495834077020 | 0.99001249583438311987 | 3.0582157e-13 | 1.65922E-10 |
| 0.2 | 0.960199733523725120 | 0.96019973353084221493 | 7.1173208e-12 | 4.76275E-10 |
| 0.3 | 0.911009467376818090 | 0.9110094674175072587 | 4.06891997e-11 | 6.23182E-10 |
| 0.4 | 0.843182982008655380 | 0.8431829821463217198 | 1.3766647e-10 | 19.9134E-10 |
| 0.5 | 0.757747685671118280 | 0.7577476860218364639 | 3.5071831e-10 | 3.28882E-10 |
| 0.6 | 0.656006844729034810 | 0.6560068454770467172 | 7.48011825e-10 | 1.27096E-09 |
| 0.7 | 0.539526561853465480 | 0.5395265632659961839 | 1.4125309e-10 | 4.84653E-09 |
| 0.8 | 0.410120128041496560 | 0.4101201304827433837 | 2.44124712e-09 | 1.09585E-08 |
| 0.9 | 0.269829904811993430 | 0.2698299832346098079 | 7.84226164e-08 | 2.01880E-08 |
| 1.0 | 0.120906917604419300 | 0.1209069013594949181 | 1.62449244e-08 | 3.53956E-08 |

Example 2

Stiff problem

$$y''' = -y, \quad y(0) = 1, y'(0) = -1, y''(0) = 1, h = \frac{1}{10}$$

Exact Solution

$$y(x) = e^{-x}$$

Source: Olabode (2013).

Table 2: Comparison of new result with results from existing methods

| X | YEX | Computed | Error in New Result | Error in Olabode (2013) |
|-----|------------------------|-------------------------|---------------------|-------------------------|
| 0.1 | 0.9048374180359597316 | 0.90483741803367739824 | 2.282175e-12 | 1.66845548E-09 |
| 0.2 | 0.81873075307798185867 | 0.81873075303378880241 | 4.419306e-11 | 1.07247812E-08 |
| 0.3 | 0.74081822068171786607 | 0.74081822050627465162 | 1.175443e-10 | 1459.00969E-08 |
| 0.4 | 0.6703200460356393007 | 0.670320045593692612750 | 4.419467e-10 | 5.60600832E-08 |
| 0.5 | 0.6065306597126334236 | 0.606530658825906540790 | 8.867269e-10 | 1.07616918E-07 |
| 0.6 | 0.54881163609402643863 | 0.548811634543275369750 | 1.550751e-10 | 1.80036532E-07 |
| 0.7 | 0.49658530379140951470 | 0.49658530131770368837 | 2.473706e-9 | 3.31228689E-06 |
| 0.8 | 0.4493289641172215914 | 0.44932896042249660546 | 3.694725e-9 | 3.46704102E-04 |
| 0.9 | 0.40656965974059911188 | 0.40656965448751866336 | 5.253081e-9 | 2.27964863E-03 |
| 1.0 | 0.36757944117144232160 | 0.36787943398259631958 | 7.188846e-9 | 6.38022004E-03 |

4.0 CONCLUSION

This paper presented a direct approach for solving third order initial value problems of ordinary differential equations. The introduction of many interpolation point enhanced order of accuracy of the developed method and gave room for the block method to adequately solve a third order initial value problem as revealed in the tables above. Hence the developed method performed better when compared with an existing work.

REFERENCES

Awoyemi, D. O. (2001). A New Sixth-order Algorithm for General Second Order Ordinary Differential Equation, *International Journal of Computer Mathematics* 77, 117-124.

Awoyemi, D. O. (2003) A P-stable linear multistep method for solving general third Order Ordinary Differential Equations, *International Journal of Computer Mathematics*, 80(8) , 985-991.

Awoyemi, D. O. & Kayode, S. J. (2005). A Maximal Order Collocation Method for Direct Solution of Initial Value Problems of General Second Order Ordinary Differential Equation, *Proceeding of the Conference Organized by the National Mathematical Centre, Abuja, Nigeria*

Adesanya, A. O., Udoh, M.O. & Alkali, A.M. (2012). A new Block Predictor-Corrector Algorithm for the Solution of $y''' = f(x, y, y', y'')$, *American Journal of Computational Mathematics* 2, 341-344.

- Jator, S. N. & Li, J. (2009). A Self Starting Linear Multistep Method for the General Second Order Initial Value Problems, *International Journal of Computer Mathematics* 86(5) 827-836.
- James, A. A., Adesanya, A. O. & Joshua, S. (2013). Continuous block method for the solution of second order initial value problems of ordinary differential equations, *International Journal of Pure and Applied Mathematics* 83(3), 405-416.
- Adesanya, A. O., Udoh, A. M. & Ajileye, A. M. (2013). A new hybrid block method for the solution of general third order initial value problems of ordinary differential equations, *International Journal of Pure and Applied Mathematics* 86(2), 37-48.
- Awoyemi, D. O., Idowu, O. M., A. (2005), Class of Hybrid Collocation Methods for Third-Order Ordinary Differential Equations, *International Journal of Computer Mathematics*, 82(10), 1287-1293.
- Kayode, S. J. & Obarhua, F. O. (2017). Symmetric 2-Step 4-Point Hybrid Method for the Solution of General Third Order Differential Equations. *Journal of Applied and Computational Mathematics*, 6,348.
- Kuboye, J. O., & Omar, Z. (2015). Numerical Solution of Third Order Ordinary Differential Equations Using a Seven Block Method. *International Journal of Mathematical Analysis Vol 9. No15*, 743-754.
- Areo, E. A. & Adeniyi, R. B. (2014). Block Implicit One-Step Method for the Numerical Integration of Initial Value Problems in Ordinary Differential Equations. *International Journal of Mathematics and Statistics Studies*, 2(3), 4-13.
- Olabode, B. T. (2013) Block Multistep Method for the Direct Solution of third Order of Ordinary Differential Equations. *FUTA Journal of Research in Sciences*, (2), 194-200.v