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**Numerical Solutions of Elliptic Partial Differential Equations**

**S. Emmanuel, H. O. Edogbanya and B. Omodara**

*Department of Mathematical Sciences, Federal University Lokoja, Kogi State, Nigeria.*

**Corresponding Authors Email address:** *helen.edogbanya@fulokoja.edu.ng*

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**ABSTRACT**

Numerical techniques for the solution of two dimensional Elliptic partial differential equations such as Laplace and Poisson equations were investigated. These types of differential equations have specific applications in physical and engineering models. The discrete approximation of both equations is based on finite difference method. In this article, five points finite difference approximation was used for two dimensional Laplace's and Poisson's equations. To solve the resulting finite difference approximation, basic iterative methods; Jacobi, Gauss-Seidel and Successive over Relaxation (SOR) have been used. Two model problems were solved and concluding remarks are presented

**Keywords:** PDEs, Boundary value problems, Jacobi, Gauss-Seidel and SOR Method.

**1.0 INTRODUCTION**

**1.1 Introduction to Problem**

The solution of PDEs is a very important topic in all areas of science and Engineering (e.g. in Heat transfer, fluid dynamics, quantum mechanics, boundary layer theory etc.).

Taking a general form of the partial differential equations (Yousif, 2014).

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0 \quad (1)$$

PDEs can be classified into the Elliptic, Hyperbolic and the Parabolic PDEs.

This classification is based on the characteristic polynomial of (1) where the quantity  $B^2 - 4AC$  is the discriminant. Below is a table showing the classification of partial differential equation based on (1) and the discriminant

$B^2 - 4AC$	Equation Type	Typical Physics Description
$< 0$	Elliptic	Steady state phenomenon
$= 0$	Parabolic	Heat Flow and Diffusion process
$> 0$	Hyperbolic	Vibrating Systems and Wave Motion

**1.2 Elliptic partial Differential Equations**

The Elliptical partial differential equation has many applications in mechanical and thermal equilibrium of bodies, fluids, mechanical and electromagnetic potentials (Yousif, 2014).

A typical example of an Elliptic partial differential equations are the Poisson and Laplace equations in two dimensions

The partial differential equation is said to be self-adjoint Elliptic if (1) can be replaced by

$$\frac{\partial}{\partial x} \left( A(x, y) \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( C(x, y) \frac{\partial u}{\partial y} \right) + Fu + G = 0 \quad (2)$$

The self-adjoint Elliptic partial differential equation (2) is said to be Poisson's equations when  $A(x, y) = C(x, y) = 1$  and  $F = 0$  which gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -G(x, y) \quad (3)$$

and equation (2) is a Laplace equation when  $G(x, y) = 0$  in (3), which is then of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (4)$$

### 1.3 Boundary Value Problems

Boundary value problems for a given differential equation consists of finding a solution of the given differential equations subject to a set of boundary conditions.

Three major types of boundary value problems are encountered in the solution of PDEs:

1. Dirichlet Boundary Conditions
2. Neuman Boundary Conditions
3. Robin Boundary Conditions

### 1.4 Finite Difference Method (FDM)

The finite difference method is based on discrete approximations of the partial derivatives of the partial differential equations obtained by Taylor Series expansion near the point of interest. [2]

Considering a function

$$u(x) = \sum_{n=0}^{\infty} h^2 \frac{u^n(x)}{n!} \quad (5)$$

The Taylors series expansion of the function (4) about a point  $x_0$  is given has

$$\begin{aligned} u(x) &= u(x_0 + h) \\ &= u(x_0) \\ &+ hu'(x_0) \\ &+ \frac{1}{2!}h^2u''(x_0) \\ &+ \frac{1}{3!}h^3u''' + \dots \end{aligned} \quad (6)$$

since (6) is valid for any increment in space (x) then we can replace h by -h

$$\begin{aligned} u(x) &= u(x_0 - h) \\ &= u(x_0) \\ &- hu'(x_0) \\ &+ \frac{1}{2!}h^2u''(x_0) \\ &- \frac{1}{3!}h^3u''' + \dots \end{aligned} \quad (7)$$

Dividing (6) and (7) by h, adding the former to the latter and neglecting terms of order two and higher we get an approximate value for  $u''(x_0)$

$$\approx \frac{u(x_0 + h) - 2u(x_0) + u(x_0 - h)}{h^2} \quad (8)$$

Equation (8) is called the central difference formula for the second derivative of  $u(x_0)$

## 2.0 METHODS

### 2.1 Iterative Method

Considering a system of  $n$  linear equations and  $m$  variable

$$\begin{aligned} u_{11}x_1 + u_{12}x_2 + \dots + u_{1n}x_n &= b_1 \\ u_{21}x_1 + u_{22}x_2 + \dots + u_{2n}x_n &= b_2 \\ \vdots &\vdots \\ u_{n1}x_1 + u_{n2}x_2 + \dots + u_{nm}x_n &= b_n \end{aligned} \quad (9)$$

where the coefficient matrix of  $u$  has no zeros on its main diagonal.

Solving the first equation of (9) for  $x_1$ , second equation in (9) for  $x_2$ , the third for  $x_3$  and so on to obtain the re-written equation

$$\begin{aligned} x_1 &= \frac{1}{u_{11}}(b_1 - u_{12}x_2 - u_{13}x_3 - \dots) \\ x_2 &= \frac{1}{u_{22}}(b_2 - u_{21}x_1 - u_{23}x_3 - \dots) \\ \vdots &= \vdots \\ x_n &= \frac{1}{u_{nn}}(b_n - u_{n1}x_1 - u_{n2}x_2 - \dots) \end{aligned} \quad (10)$$

Considering 3 Iterative Methods

### 2.2 The Gauss-Jacobi Method

Equation (10) can be written in matrix form as  $(D - L - u)x = bm$

Where:

$D$  is the upper triangular matrix

$L$  is the lower triangular matrix

$U$  is the upper triangular matrix

$m$  is non-singular and easily invertible matrix

$u$  is unknown

Then

$$x = D^{-1}(L + U)x + D^{-1}b$$

The Jacobi iterative method expresses the  $(n + 1)$  iterative values in exclusively terms of the  $n^{th}$  values corresponding to the equation (10)

$$\begin{aligned} x_1^{(n+1)} &= \frac{1}{u_{11}}(b_1 - u_{12}x_2^{(n)} - u_{13}x_3^{(n)} - \dots - u_{1n}x_n^{(n)}) \\ x_2^{(n+1)} &= \frac{1}{u_{22}}(b_2 - u_{21}x_1^{(n)} - u_{23}x_3^{(n)} - \dots - u_{2n}x_n^{(n)}) \\ &\vdots \\ &= \frac{1}{u_{nn}}(b_n - u_{n1}x_1^{(n)} - u_{n2}x_2^{(n)} - \dots - u_{nm-1}x_{m-1}^{(n)}) \end{aligned}$$

The General Case of the Jacobi iterative method is given by

$$\begin{aligned} x_i^{(n+1)} &= \frac{1}{u_{ii}} \left\{ b_i - \sum_{j=1}^{i-1} u_{ij}x_j^{(n)} - \sum_{j=i+1}^m u_{ij}x_j^{(n)} \right\}, \quad i \\ &= 1(1)m \end{aligned} \tag{11}$$

where  $x_i^{(1)}$  denotes the first approximation to  $x_i$ ,  $x_2^{(n)}$  denotes the second approximation for  $x_2$  and so on.

Applying the Gauss-Jacobi to the five-point finite difference approximations for the Laplace and Poisson's Equation respectively we get

$$u_{i,j}^{n+1} = \frac{u_{i+1,j}^{(n)} + u_{i-1,j}^{(n)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n)}}{4} \tag{12}$$

and

$$u_{i,j}^{n+1} = \frac{u_{i+1,j}^{(n)} + u_{i-1,j}^{(n)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n)} - h^2 f_{i,j}}{4} \tag{13}$$

**2.3 The Gauss-Seidel Method**

With the Gauss-Jacobi Method, the values of  $x_i$  obtained in the  $n$ th approximation remains unchanged until the entire  $(n + 1)$ th approximation has been calculated. Gauss-Seidel method however uses new values of  $x_i$  as soon as they are known.

The general case of the Gauss-Seidel is given by

$$\begin{aligned} x_i^{(n+1)} &= \frac{1}{u_{ii}} \left\{ b_i - \sum_{j=1}^{i-1} u_{ij}x_j^{(n+1)} - \sum_{j=i+1}^m u_{ij}x_j^{(n)} \right\}, \quad i \\ &= 1(1)m \end{aligned} \tag{14}$$

Applying this method to the five-point finite difference approximations for the Laplace and Poisson equations respectively,

we get

$$u_{i,j}^{n+1} = \frac{u_{i+1,j}^{(n+1)} + u_{i-1,j}^{(n+1)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n)}}{4} \tag{15}$$

and

$$u_{i,j}^{n+1} = \frac{u_{i+1,j}^{(n+1)} + u_{i-1,j}^{(n+1)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n)} - h^2 f_{i,j}}{4} \tag{16}$$

**2.4 Successive Over-Relaxation Method (SOR)**

The General Case of the SOR method is given by

$$\begin{aligned} x_i^{(n+1)} &= (1 - \omega) \cdot u_{i,j}^{(n)} + \frac{1}{u_{ii}} \left\{ b_i - \sum_{j=1}^{i-1} u_{ij}x_j^{(n+1)} - \sum_{j=i+1}^m u_{ij}x_j^{(n)} \right\}, \quad i = 1(1)m \end{aligned} \tag{17}$$

where  $\omega$  is the called the acceleration parameter or relaxation factor which generally lies in the range  $1 < \omega < 2$  and  $\omega = 1$  gives the Gauss-Seidel method.

The application to the five-point finite difference approximation of the Laplace and Poisson's equations gives

$$u_{i,j}^{n+1} = (1 - \omega) \cdot u_{i,j}^{(n)} + \frac{u_{i+1,j}^{(n+1)} + u_{i-1,j}^{(n+1)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n)}}{4}$$

and

$$u_{i,j}^{n+1} = (1 - \omega) \cdot u_{i,j}^{(n)} + \frac{u_{i+1,j}^{(n+1)} + u_{i-1,j}^{(n+1)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n)} - h^2 f_{i,j}}{4}$$

### 2.5 Rate of Convergence

**Theorem 1:** if  $A = M - N$  is a regular splitting of the matrix  $A$  and  $A^{-1} \geq 0$  then

$$\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1 + \rho(A^{-1}N)} \leq 1$$

Thus, the matrix  $M^{-1}N$  is convergent and the iterative method of equation (9) converges for any initial vector of  $u^{(0)}$ . (Saad, 2003).

**Proof:**

Assume that  $A$  is non-singular with  $A^{-1} \geq 0$ , then since

$$A = M - N \Rightarrow M = A + N = N(I + N^{-1}A)$$

$$M = N(I + N^{-1}A) \Rightarrow N^{-1}M = (I + N^{-1}A)$$

$$(N^{-1}M)^{-1} = (I + N^{-1}A)^{-1} \Rightarrow$$

$$M^{-1}N = (I + N^{-1}A)^{-1}$$

$$M^{-1}N = [N^{-1}(N + A)]^{-1} = (N + A)^{-1}N$$

$$M^{-1}N = [A(A^{-1}N + I)]^{-1}N$$

$$M^{-1}N = [A^{-1}N + I]^{-1}A^{-1}N$$

$$M^{-1}N = [A^{-1}N + I]^{-1}A^{-1}N$$

Let  $G = A^{-1}N$  then

$$M^{-1} = (G + 1)^{-1}G$$

Because of this relation, if  $x$  is an eigenvector of  $G$  corresponding to the eigenvalue  $\lambda$  then  $1 + \lambda \neq 0$

$$Gx = \lambda x \quad \text{and} \quad (G + 1)^{-1}Gx = \frac{\lambda}{1 + \lambda}x$$

It follows that  $x$  is also an eigenvector of  $M^{-1}N$  corresponding to the Eigen value of  $\mu$  given by

$$(M^{-1}N)x = \mu x = (G + 1)^{-1}Gx = \frac{\lambda}{1 + \lambda}x$$

$$\Rightarrow \mu = \frac{\lambda}{1 + \lambda}$$

We can maximize  $\mu$  by choosing  $\lambda = \rho(G)$  then

$$\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1 + \rho(A^{-1}N)} < 1$$

**Theorem 2:** if a matrix  $A$  is strictly diagonally dominant or irreducible diagonally dominant, the associated Jacobi and Gauss-Seidel Iterations converge for any  $u^{(0)}$ . (Axelsson 1994).

**Theorem 3:** suppose that the matrix  $A = D - L - U$  has the following properties for any  $c \in \mathbb{R}$  then

$$\det(cD - L - U) = \det(cD - \delta L - \delta^{-1}U)$$

for all  $\delta \in \mathbb{R} \setminus \{0\}$ . Then the following properties holds:

if  $\gamma \neq 0$  is an eigen value of  $G_\omega$  for some  $\omega \in (0,2)$  then

$$\mu = \frac{\gamma + \omega - 1}{\omega \gamma^2} \quad (\text{Saad, 2003}).$$

**Theorem 4:** if  $A$  is consistently ordered then

$$\rho(G_{GS}) - [\rho(G_J)]^2$$

that is, Gauss-Seidel iterations converges twice as fast as Jacobi Iterations. (Richard et al. 2010)

**Theorem 5:** Suppose that  $A$  is consistently ordered and that  $G$  has only real eigenvalues and that  $\beta = \rho(G) < 1$ . Then SOR iteration converges twice for every  $\omega \in (0,2)$  and the spectral radius of the matrix is

$$\rho(G_\omega) = \begin{cases} 1 - \omega + \frac{1}{2}\omega^2\beta^2 + \omega\beta\sqrt{1 - \omega + \frac{\omega^2\beta^2}{4}} & \text{for } 0 < \omega < \omega_{opt} \\ \omega - 1 & \text{for } \omega_{opt} \leq \omega < 2 \end{cases}$$

where

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \beta^2}} \quad (\text{Saad, 2003})$$

### 3.0 NUMERICAL IMPLEMENTATIONS OF THE METHODS

Problem 1: Laplace Equation

$$u_{xx} + u_{yy} = e^{-xy}(x^2 + y^2)$$

$$u(x, 1) = \sin\pi x \text{ for } 0 \leq x \leq 1$$

$$u(x, 1) = e^{-\pi} \sin\pi x \text{ for } 0 \leq x \leq 1$$

$$u(0, y) = 0 \leq y \leq 1$$

$$u(1, y) = 0 \text{ for } 0 \leq y \leq 1$$

Exact:  $u(x, y) = e^{-xy}$

$$u(0, y) = 0 \text{ for } 0 \leq y \leq 1$$

$$u(1, y) = 0 \text{ for } 0 \leq y \leq 1$$

Exact:  $u(x, y) = e^{-\pi y}$

Problem 2: Poisson Equation

$$u_{xx} + u_{yy} = 0$$

$$u(x, 0) = \sin \pi x \text{ for } 0 \leq x \leq 1$$

$$u(x, 1) = e^{-\pi} \sin \pi x \text{ for } 0 \leq x \leq 1$$

The iterative algorithms used in solving problem 1 and 2 was done by the use of Matlab with mesh size of  $h = 0.1$  and a stopping criteria or tolerance of

$$\frac{\|u^{(k)} - u^{(k-1)}\|}{u^{(k)}} < 10^{-5}$$

**Table 1: Exact Solution of the Laplace Equation**

	$i = 0$	1	2	3	4	5	6	7	8	9	1
$x = 0$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	
$j = 0, y = 0$	0	0.3090	0.5878	0.8090	0.9511	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000
$j = 0, y = 0$	0	0.2257	0.4293	0.5909	0.6947	0.7304	0.6947	0.5909	0.4293	0.2257	0.0000
$j = 0, y = 0$	0	0.1649	0.3136	0.4316	0.5074	0.5335	0.5074	0.4316	0.3136	0.1649	0.0000
$j = 0, y = 0$	0	0.1204	0.2290	0.3152	0.3706	0.3897	0.3706	0.3152	0.2290	0.1204	0.0000
$j = 0, y = 0$	0	0.0879	0.1673	0.2303	0.2707	0.2846	0.2707	0.2303	0.1673	0.0879	0.0000
$j = 0, y = 0$	0	0.0642	0.1222	0.1682	0.1977	0.2079	0.1977	0.1682	0.1222	0.0642	0.0000
$j = 0, y = 0$	0	0.0469	0.0892	0.1228	0.1444	0.1518	0.1444	0.1228	0.0892	0.0469	0.0000
$j = 0, y = 0$	0	0.0343	0.0652	0.0897	0.1055	0.1109	0.1055	0.0897	0.0652	0.0343	0.0000
$j = 0, y = 0$	0	0.0250	0.0476	0.0655	0.0770	0.0810	0.0770	0.0655	0.0476	0.0250	0.0000
$j = 0, y = 0$	0	0.0183	0.0348	0.0479	0.0563	0.0592	0.0563	0.0479	0.0348	0.0183	0.0000
$j = 0, y = 0$	0	0.0134	0.0254	0.0350	0.0411	0.0432	0.0411	0.0350	0.0254	0.0134	0.0000

**Table 2: Numerical Solutions of Laplace Equation using the Jacobi-Method**

	$i = 0$	1	2	3	4	5	6	7	8	9	1
$x = 0$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.
$j = 0, y = 0$	0	0.3090	0.5878	0.8090	0.9511	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000
$j = 0, y = 0.1$	0	0.2263	0.4304	0.5924	0.6964	0.7322	0.6964	0.5924	0.4304	0.2263	0
$j = 0, y = 0.2$	0	0.1657	0.3151	0.4337	0.5099	0.5361	0.5099	0.4337	0.3151	0.1657	0
$j = 0, y = 0.3$	0	0.1213	0.2307	0.3176	0.3733	0.3925	0.3733	0.3176	0.2307	0.1213	0
$j = 0, y = 0.4$	0	0.0888	0.1689	0.2325	0.2733	0.2873	0.2733	0.2325	0.1689	0.0888	0
$j = 0, y = 0.5$	0	0.0650	0.1236	0.1701	0.2000	0.2103	0.2000	0.1701	0.1236	0.0650	0
$j = 0, y = 0.6$	0	0.0475	0.0904	0.1245	0.1463	0.1538	0.1463	0.1245	0.0904	0.0475	0
$j = 0, y = 0.7$	0	0.0347	0.0661	0.0910	0.1069	0.1124	0.1069	0.0910	0.0661	0.0347	0
$j = 0, y = 0.8$	0	0.0254	0.0482	0.0664	0.0780	0.0821	0.0780	0.0664	0.0482	0.0254	0
$j = 0, y = 0.9$	0	0.0185	0.0351	0.0483	0.0568	0.0597	0.0568	0.0483	0.0351	0.0185	0
$j = 10, y = 1.0$	0	0.0134	0.0254	0.0350	0.0411	0.0432	0.0411	0.0350	0.0254	0.0134	0.0000

**Table 3: Numerical Solution of the Laplace Equation Using the Gauss-Seidel Method**

$i = 0$ $x = 0$	1	2	3	4	5	6	7	8	9	10
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.
$j = 0, y = 0$ 0.0000	0	0.3090	0.5878	0.8090	0.9511	1.0000	0.9511	0.8090	0.5878	0.3090
$j = 1, y = 0.1$ 0	0	0.2263	0.4304	0.5924	0.6964	0.7322	0.6964	0.5924	0.4304	0.2263
$j = 2, y = 0.2$ 0	0	0.1657	0.3151	0.4338	0.5099	0.5362	0.5099	0.4338	0.3151	0.1657
$j = 3, y = 0.3$ 0	0	0.1213	0.2307	0.3176	0.3733	0.3925	0.3733	0.3176	0.2307	0.1213
$j = 4, y = 0.4$ 0	0	0.0888	0.1689	0.2325	0.2733	0.2874	0.2733	0.2325	0.1689	0.0888
$j = 5, y = 0.5$ 0	0	0.0650	0.1236	0.1702	0.2000	0.2103	0.2000	0.1702	0.1236	0.0650
$j = 6, y = 0.6$ 0	0	0.0475	0.0904	0.1245	0.1463	0.1539	0.1463	0.1245	0.0904	0.0475
$j = 7, y = 0.7$ 0	0	0.0348	0.0661	0.0910	0.1070	0.1125	0.1070	0.0910	0.0661	0.0348
$j = 8, y = 0.8$ 0	0	0.0254	0.0482	0.0664	0.0781	0.0821	0.0781	0.0664	0.0482	0.0254
$j = 9, y = 0.9$ 0	0	0.0185	0.0351	0.0483	0.0568	0.0597	0.0568	0.0483	0.0351	0.0185
$j = 10, y = 1.0$ 0.000	0	0.0134	0.0254	0.0350	0.0411	0.0432	0.0411	0.0350	0.0254	0.0134

**Table 4: Numerical solution of the Laplace Equation Using the SOR Method**

	$i = 0$	1	2	3	4	5	6	7	8	9	10	
$x = 0$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.		
$j = 0, y = 0$	0	0.3090	0.5878	0.8090	0.9511	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000	0.0000
$j = 1, y = 0.1$	0	0.2263	0.4304	0.5924	0.6964	0.7322	0.6964	0.5924	0.4304	0.2263	0	0
$j = 2, y = 0.2$	0	0.1657	0.3151	0.4338	0.5099	0.5362	0.5099	0.4338	0.3151	0.1657	0	0
$j = 3, y = 0.3$	0	0.1213	0.2307	0.3176	0.3733	0.3925	0.3733	0.3176	0.2307	0.1213	0	0
$j = 4, y = 0.4$	0	0.0888	0.1689	0.2325	0.2733	0.2874	0.2733	0.2325	0.1689	0.0888	0	0
$j = 5, y = 0.5$	0	0.0650	0.1236	0.1702	0.2000	0.2103	0.2000	0.1702	0.1236	0.0650	0	0
$j = 6, y = 0.6$	0	0.0475	0.0904	0.1245	0.1463	0.1539	0.1463	0.1245	0.0904	0.0475	0	0
$j = 7, y = 0.7$	0	0.0348	0.0661	0.0910	0.1070	0.1125	0.1070	0.0910	0.0661	0.0348	0	0
$j = 8, y = 0.8$	0	0.0254	0.0482	0.0664	0.0781	0.0821	0.0781	0.0664	0.0482	0.0254	0	0
$j = 9, y = 0.9$	0	0.0185	0.0351	0.0483	0.0568	0.0597	0.0568	0.0483	0.0351	0.0185	0	0
$j = 10, y = 1.0$	0	0.0134	0.0254	0.0350	0.0411	0.0432	0.0411	0.0350	0.0254	0.0134	0.0000	0.0000

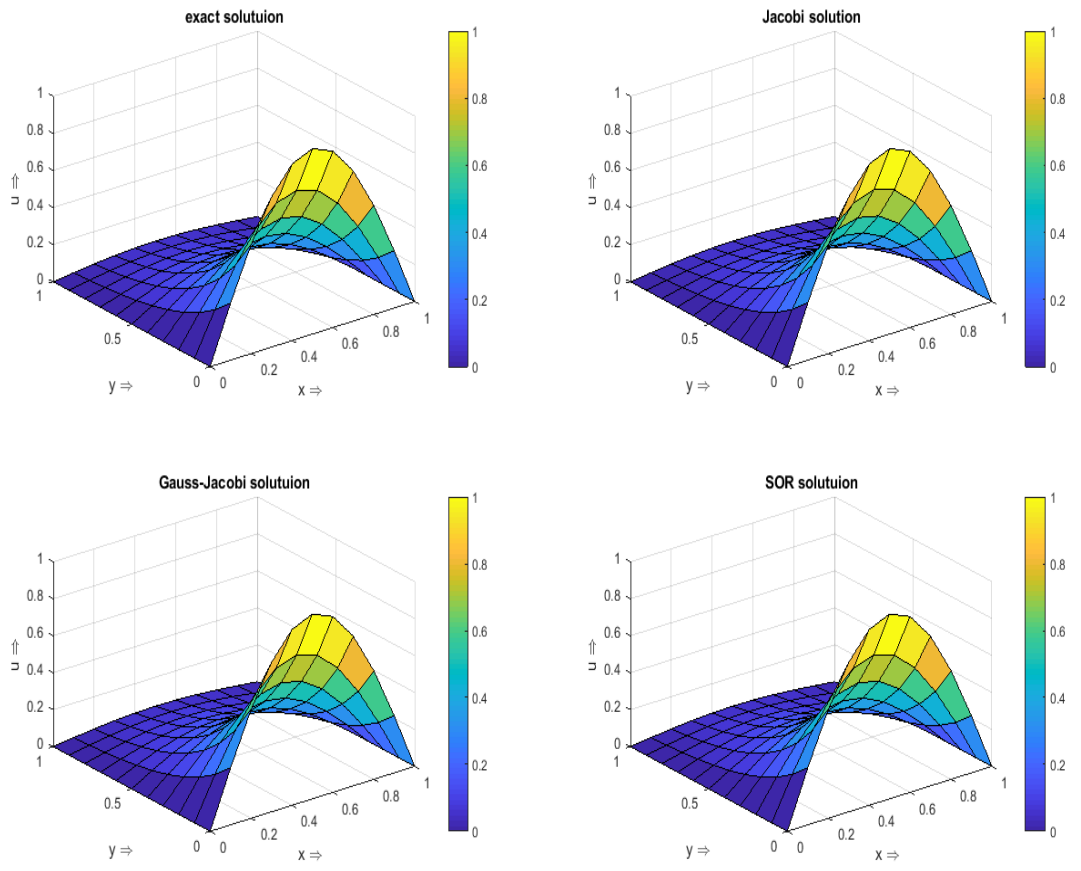


Figure 1: Graphical Representations of Laplace Equation

**Table 5: Exact Solution of the Poison Equation**

	$i = 0$	1	2	3	4	5	6	7	8	9	
10	$x = 0$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$j = 0, y = 0$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$j = 1, y = 0.1$	1.0000	0.9915	0.9841	0.9765	0.9687	0.9608	0.9525	0.9435	0.9334	0.9212	0.9048
$j = 0, y = 0.2$	1.0000	0.9835	0.9683	0.9531	0.9377	0.9220	0.9057	0.8883	0.8689	0.8464	0.8187
$j = 0, y = 0.3$	1.0000	0.9756	0.9527	0.9299	0.9070	0.8838	0.8600	0.8349	0.8076	0.7768	0.7408
$j = 0, y = 0.4$	1.0000	0.9678	0.9371	0.9067	0.8765	0.8463	0.8156	0.7837	0.7498	0.7125	0.6703
$j = 0, y = 0.5$	1.0000	0.9597	0.9213	0.8834	0.8461	0.8092	0.7723	0.7347	0.6954	0.6532	0.6065
$j = 0, y = 0.6$	1.0000	0.9514	0.9049	0.8595	0.8153	0.7722	0.7298	0.6874	0.6440	0.5983	0.5488
$j = 0, y = 0.7$	1.0000	0.9425	0.8875	0.8344	0.7835	0.7346	0.6873	0.6410	0.5948	0.5472	0.4966
$j = 0, y = 0.8$	1.0000	0.9324	0.8683	0.8072	0.7496	0.6953	0.6439	0.5947	0.5469	0.4990	0.4493
$j = 0, y = 0.9$	1.0000	0.9205	0.8460	0.7766	0.7124	0.6531	0.5982	0.5472	0.4990	0.4528	0.4066
$j = 0, y = 1.0$	1.0000	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066	0.3679

**Table 6: Numerical Solution of the Poison Equation using the Jacobi Method**

	$i = 0$	1	2	3	4	5	6	7	8	9	
10	$x = 0$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	
1.0											
$j = 0, y = 0$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
$j = 0, y = 0$	1.0000	0.9917	0.9842	0.9766	0.9689	0.9609	0.9526	0.9436	0.9334	0.9212	0.9048
$j = 0, y = 0$	1.0000	0.9842	0.9688	0.9534	0.9380	0.9222	0.9058	0.8884	0.8690	0.8465	0.8187
$j = 0, y = 0$	1.0000	0.9766	0.9534	0.9304	0.9074	0.8841	0.8602	0.8351	0.8077	0.7769	0.7408
$j = 0, y = 0$	1.0000	0.9689	0.9380	0.9074	0.8770	0.8466	0.8159	0.7840	0.7500	0.7126	0.6703
$j = 0, y = 0$	1.0000	0.9609	0.9222	0.8841	0.8466	0.8096	0.7726	0.7349	0.6956	0.6533	0.6065
$j = 0, y = 0$	1.0000	0.9526	0.9058	0.8602	0.8159	0.7726	0.7301	0.6876	0.6441	0.5983	0.5488
$j = 0, y = 0$	1.0000	0.9436	0.8884	0.8351	0.7840	0.7349	0.6876	0.6412	0.5949	0.5472	0.4966
$j = 0, y = 0$	1.0000	0.9334	0.8690	0.8077	0.7500	0.6956	0.6441	0.5949	0.5470	0.4991	0.4493
$j = 0, y = 0$	1.0000	0.9212	0.8465	0.7769	0.7126	0.6533	0.5983	0.5472	0.4991	0.4528	0.4066
$j = 0, y = 0$	1.0000	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066	0.3679

**Table 7: Numerical Solution of the Poison Equation using the Gauss-Seidel Method**

	$i = 0$	1	2	3	4	5	6	7	8	9
10	$x = 0$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1.0										
$j = 0, y = 0$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$j = 1, y = 0.1$	1.0000	0.9909	0.9838	0.9764	0.9688	0.9609	0.9526	0.9436	0.9334	0.9212
$j = 2, y = 0.2$	1.0000	0.9838	0.9685	0.9533	0.9379	0.9222	0.9058	0.8884	0.8690	0.8465
$j = 3, y = 0.3$	1.0000	0.9764	0.9533	0.9303	0.9073	0.8841	0.8602	0.8351	0.8077	0.7769
$j = 4, y = 0.4$	1.0000	0.9688	0.9379	0.9073	0.8770	0.8466	0.8159	0.7840	0.7500	0.7126
$j = 5, y = 0.5$	1.0000	0.9609	0.9222	0.8841	0.8466	0.8096	0.7727	0.7350	0.6956	0.6533
$j = 6, y = 0.6$	1.0000	0.9526	0.9058	0.8602	0.8159	0.7727	0.7301	0.6876	0.6441	0.5984
$j = 7, y = 0.7$	1.0000	0.9436	0.8884	0.8351	0.7840	0.7350	0.6876	0.6413	0.5949	0.5472
$j = 8, y = 0.8$	1.0000	0.9334	0.8690	0.8077	0.7500	0.6956	0.6441	0.5949	0.5470	0.4991
$j = 9, y = 0.9$	1.0000	0.9212	0.8465	0.7769	0.7126	0.6533	0.5984	0.5472	0.4991	0.4528
$j = 10, y = 1.0$	1.0000	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066

**Table 8: Numerical Solution of the Poison Equation using the SOR Method**

	$i = 0$	1	2	3	4	5	6	7	8	9
10	$x = 0$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1.0										
$j = 0, y = 0$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$j = 1, y = 0.1$	1.0000	0.9909	0.9838	0.9764	0.9688	0.9609	0.9526	0.9436	0.9334	0.9212
$j = 2, y = 0.2$	1.0000	0.9838	0.9685	0.9533	0.9379	0.9222	0.9058	0.8884	0.8690	0.8465
$j = 3, y = 0.3$	1.0000	0.9764	0.9533	0.9303	0.9073	0.8841	0.8602	0.8351	0.8077	0.7769
$j = 4, y = 0.4$	1.0000	0.9688	0.9379	0.9073	0.8770	0.8466	0.8159	0.7840	0.7500	0.7126
$j = 5, y = 0.5$	1.0000	0.9609	0.9222	0.8841	0.8466	0.8096	0.7727	0.7350	0.6956	0.6533
$j = 6, y = 0.6$	1.0000	0.9526	0.9058	0.8602	0.8159	0.7727	0.7301	0.6876	0.6441	0.5984
$j = 7, y = 0.7$	1.0000	0.9436	0.8884	0.8351	0.7840	0.7350	0.6876	0.6413	0.5949	0.5472
$j = 8, y = 0.8$	1.0000	0.9334	0.8690	0.8077	0.7500	0.6956	0.6441	0.5949	0.5470	0.4991
$j = 9, y = 0.9$	1.0000	0.9212	0.8465	0.7769	0.7126	0.6533	0.5984	0.5472	0.4991	0.4528
$j = 10, y = 1.0$	1.0000	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066

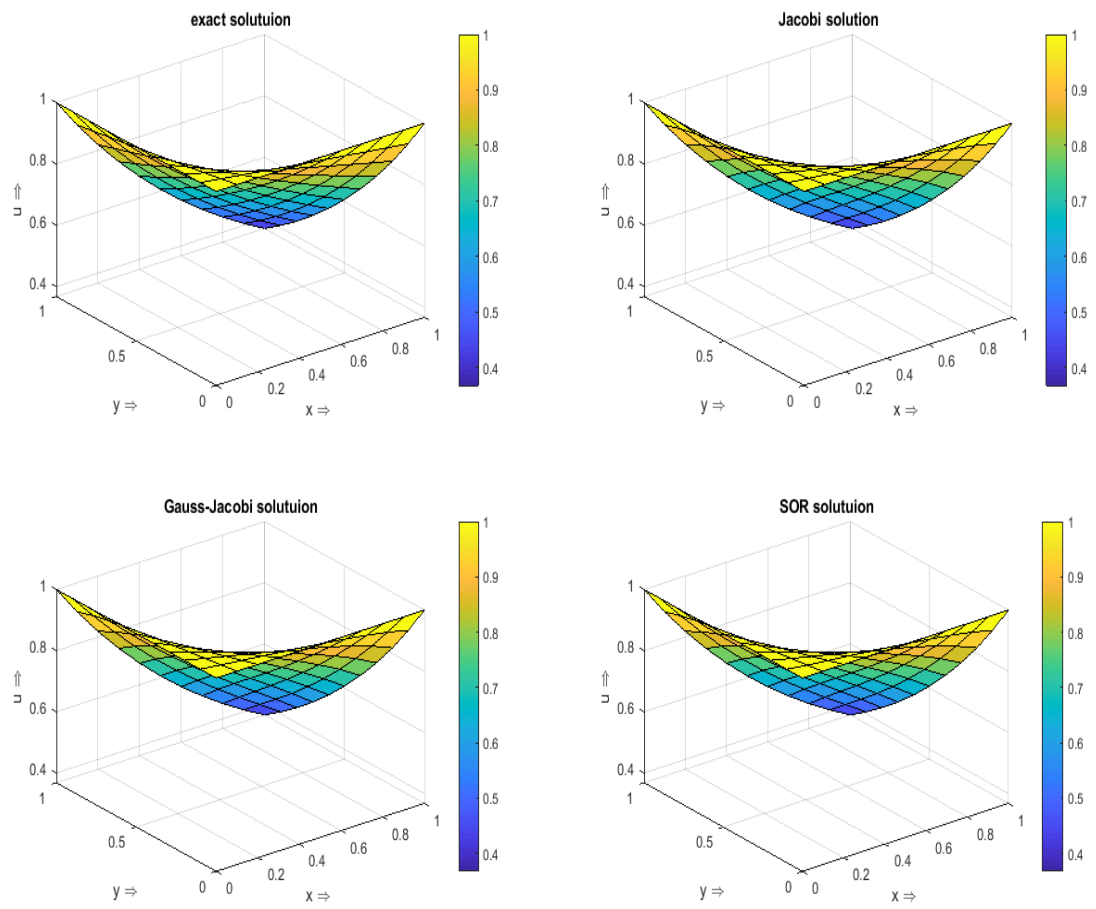


Figure 2: Graphical Representations of the Poisson Equation

#### 4.0 DISCUSSION OF RESULTS

From the analysis of the result, the Gauss Seidel method takes a fewer number of iterations to converge to the given tolerance than the Jacobi method and chanced the Gauss Seidel method has higher convergence rate than the Jacobi method. And even faster method is successive over relaxation method.

#### 5.0 CONCLUSION

The finite difference method was reviewed and emphasis were made on numerical solution of the second order two dimensional elliptic partial differential equation using the five point finite difference discretization, the generated linear system was solved by using the following ; Jacobi, Gauss Seidel and SOR.

Two practical problems were solved using MATLAB and comparison were made between the numerical and exact solutions.

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