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# The Burr X-Power Lomax Distribution: Properties, Inference, and Applications in Reliability Analysis

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***ABSTRACT***

This study proposes a novel generalized probability distribution, referred to as the **Burr X–Power Lomax Distribution (BPLD)**, by incorporating the Burr X-G family into the Power Lomax distribution. The new distribution enhances the flexibility of the conventional Lomax and Power Lomax models, making it appropriate for analyzing skewed and heavy-tailed data frequently observed in reliability, engineering, survival, and lifetime studies. Key statistical properties of the BPLD are derived, including its probability density function, cumulative distribution function, quantile function, ordinary moments, moment generating function, characteristic function, survival function, and hazard rate function. The model parameters are estimated using the maximum likelihood estimation (MLE) technique. The theoretical findings demonstrate that the proposed distribution is highly flexible and provides an effective alternative for modeling lifetime and reliability data with varying distributional characteristics.

**Keywords:** Burr X–Power Lomax Distribution (BPLD), generalized Lomax distribution, Burr X-G family of distributions, maximum likelihood estimation (MLE), lifetime data analysis, reliability modeling.

## 1.0 INTRODUCTION

The properties of probability distribution functions and their underlying relationships are essential for accurately modeling real-world phenomena across various fields of study. For several decades, many probability distributions have served well in various branches of *engineering*, finance, biological, environmental, economics, and medical sciences, to name a few. When the function or distribution that is assumed does not fit the data well enough, one common solution is to use a compound distribution or a more flexible family of distributions. As a result, many probability distribution families and generalized probability distributions have been found useful and appropriate for modelling real-life datasets.

Several generalized families of probability distributions have been developed to enhance the flexibility of standard distributions for modeling diverse types of data. Notable examples include the truncated Burr X-G family of distributions proposed by Bantan *et al.* (2021), the flexible Burr X-G family by Al-Babtain et al. (2021), the shifted Gompertz-G family by Eghwerido *et al.* (2021), and the flexible odd Kappa-G family introduced by Al-Shomrani and Al-Arfaj (2021). Other notable contributions include the truncated Cauchy power Weibull-G class of distributions by Alotaibi et al. (2022), the odd Perks-G class by Elbatal et al. (2022), the Marshall–Olkin odd *et al* power generalized Weibull-G family by Chipepa. (2022), the shifted exponential-G family by Eghwerido *et al*. (2022), and the odd Chen-G family proposed by Anzagra et al. (2022). More recent developments include the sine family of generalized distributions introduced by Benchiha et al. (2023), the Topp–Leone Type II exponentiated half logistic-G family by Gabanakgosi and Oluyede (2023), the bivariate Lomax-G family by Fayomi et al. (2023), the X-exponential-G family by Mohammad (2024), and the Fréchet-G family of continuous probability distributions proposed by Ieren et al. (2024). families have been used to study compound distributions such as the exponential-Lindley distribution by Ieren and Balogun (2021), the transmuted odd Lindley-Rayleigh distribution by Umar *et al.* (2021), the Sine-exponential distribution by Isa et al. (2022a), the Lindley-Lomax distribution by Agbor *et al.* (2022), the sine Burr XII distribution by Isa *et al.* (2022b), the sine-Lomax distribution by Mustapha et al. (2023), the Chen-Perks distribution by Mendez-Gonzalez et al. (2023), and the sine Lomax-exponential distribution by Joel et al. (2024).

The **Lomax distribution**, also known as the **Pareto Type II distribution**, was first introduced by Lomax (1954) for modeling business failure data. Since its introduction, it has been widely applied in various fields, including income and wealth distribution, city size analysis, actuarial science, medical and biological research, engineering, as well as lifetime and reliability studies. A random variable (X) is said to follow a Lomax distribution with parameters $\alpha$ and $\beta$ if its probability density function (pdf) is defined as follows:

$$g(x)=\frac{\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)}$$

(1)

The corresponding cumulative distribution function (CDF) is expressed as follows:

$$G(x)=1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}$$

(2)

For $x>0,\alpha>0,\beta>0$ where $\alpha$ and $\beta$ are the shape and scale parameters respectively.

Recently, a new extension of the Lomax distribution has been proposed in the literature by considering the power transformation $X=T^{\frac{1}{\lambda}}$, where the random variable $T$ is said to have follow the Lomax distribution with parameters $\alpha$ and $\beta$. The distribution of $X$ according to Rady *et al.* (2016) is referred to as Power Lomax distribution.

The *pdf* of the Power Lomax distribution is defined by

$$g(x)=\alpha\lambda\beta^{\alpha}x^{\lambda-1}\left(\beta+x^{\lambda}\right)^{-(\alpha+1)},x>0,\alpha,\lambda,\beta>0 \quad (3)$$

The corresponding cumulative distribution function (CDF) of Power Lomax distribution is given by

$$G(x)=1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha},x>0,\alpha,\lambda,\beta>0$$

(4)

where, $x>0,\alpha>0,\beta>0,\lambda>0$ $\alpha$ and $\beta$ are the shape and scale parameters respectively and $\lambda$ is the power parameter.

Therefore, the target of this paper is to develop a new extension of the Power Lomax distribution using the Burr X-G family of distributions.

The rest of this paper is presented as follows: the Burr X-Power Lomax distribution (BPLD) and its plots are given in section 2. A simplification of the PDF is given in section 3. Section 4 presents the properties of the BPLD such as quantile function, moments and generating functions. The estimation of parameters using maximum likelihood estimation (MLE) is contained in section 5 Applications of the BPLD to some real-life datasets is contained in section 6 and the conclusion of the study is given in section 7.

## 2.0 THE BURR X-POWER LOMAX DISTRIBUTION (BPLD)

Yousof et al. (2017) introduced the Burr X-G family of distributions as a flexible framework for constructing compound distributions using the Burr X distribution as a generator. Let (G(x)) and (g(x)) denote the cumulative distribution function (cdf) and probability density function (pdf), respectively, of any continuous baseline distribution. Then, the cumulative distribution function (cdf) of the Burr X-G family of distributions is given by

$$F(x)=\left\{1-e^{-\left[\frac{G(x)}{1-G(x)}\right]^{2}}\right\}^{\theta} \quad (5)$$

The corresponding probability density function (pdf) of the Burr X-G family of distributions is expressed as follows:

$$f(x)=\frac{2\theta g(x)G(x)}{\left[1-G(x)\right]^{3}}e^{-\left[\frac{G(x)}{1-G(x)}\right]^{2}}\left(1-e^{-\left[\frac{G(x)}{1-G(x)}\right]^{2}}\right)^{\theta-1} \quad (6)$$

where $g(x)$ and $G(x)$ are the pdf and the cdf of any continuous distribution to be modified respectively and $\theta>0$ is the extra shape parameter of the Burr X-G family of distribution responsible for the flexibility in the new compound distribution.

Using equation (3) and (4) in (5) and (6) and simplifying, we obtain the cdf and pdf of the Burr X-power Lomax distribution (BPLD) as follows:

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$$F(x;\theta,\xi)=\left\{1-e^{-\left[\frac{G(x;\xi)}{1-G(x;\xi)}\right]^{2}}\right\}^{\theta}$$

$$F(x;\alpha,\lambda,\beta,\theta)=\left\{1-e^{-\left[\frac{1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}}{1-\left[1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right]}\right]^{2}}\right\}^{\theta} \qquad (7)$$

And

$$f(x;\theta,\xi)=\frac{2\theta g(x;\xi)G(x;\xi)}{\left[1-G(x;\xi)\right]^{3}}e^{-\left[\frac{G(x;\xi)}{1-G(x;\xi)}\right]^{2}}\left(1-e^{-\left[\frac{G(x;\xi)}{1-G(x;\xi)}\right]^{2}}\right)^{\theta-1} \qquad (8)$$

respectively.
For ($x > 0$), where α, $\beta$ , and θ are the shape parameters and λ is the scale parameter, Equations (7) and (8) define the cumulative distribution function (CDF) and probability density function (PDF) of the Burr X-Power Lomax Distribution (BPLD), respectively.
To demonstrate the flexibility of the BPLD, the PDF and CDF are plotted for selected values of the parameters α, $\beta$ and λ. The corresponding plots, presented in the figures below, illustrate the various shapes that the distribution can exhibit under different parameter settings.

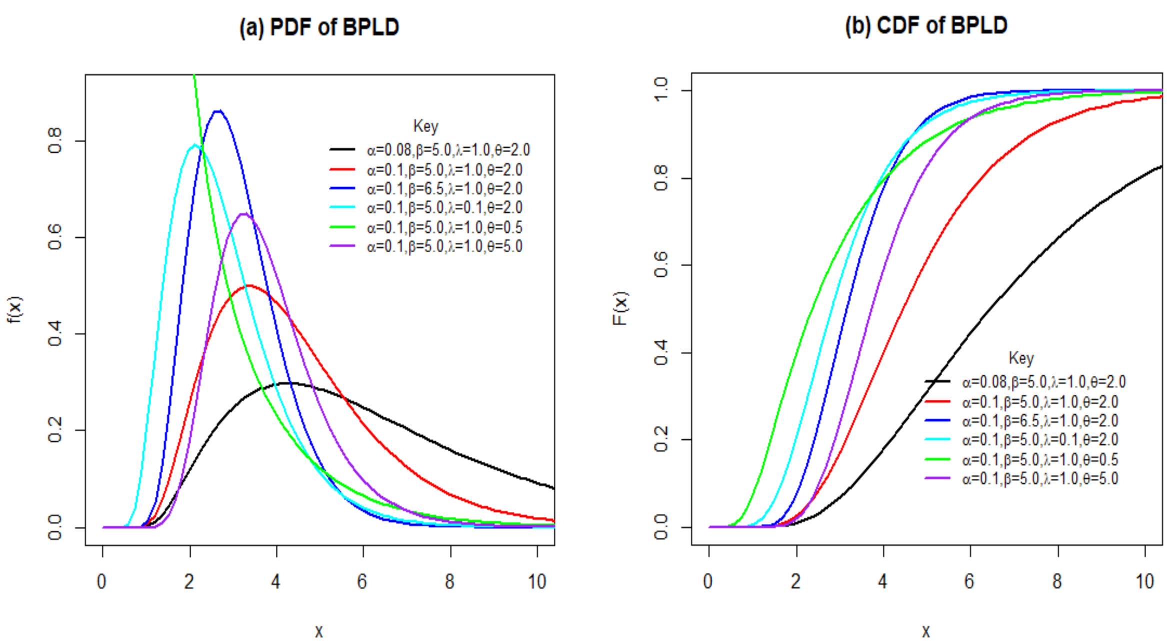


**Figure 1**: PDF and CDF of the BPLD.

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Figure 1(a) illustrates that the BPLD can exhibit a variety of shapes, including left-skewed and right-skewed forms, depending on the values of its parameters. This flexibility makes the distribution suitable for modeling datasets with diverse distributional characteristics. Furthermore, Figure 1(b) shows that the cumulative distribution function (CDF) increases monotonically as (x) increases and gradually approaches 1 for large values of (x), which is consistent with the expected behavior of a valid cumulative distribution function.

## 3.0 SIMPLIFICATION OF THE PDF OF THE BPLD.

This section presents a useful series expansion and a mixture (linear) representation of the probability density function (PDF) of the Burr X-Power Lomax Distribution (BPLD). These representations facilitate the derivation of several statistical properties of the distribution. The PDF of the BPLD is given by:

$$f(x)=\frac{2\theta\alpha\lambda\beta^{\alpha}x^{\lambda-1}\left(\beta+x^{\lambda}\right)^{-(\alpha+1)}\left(1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)}{\left[1-\left(1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)\right]^{3}e^{\left[\frac{1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}}{1-\left(1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)}\right]^{2}}}\left(1-e^{-\left[\frac{1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}}{1-\left(1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)}\right]^{2}}\right)^{\theta-1} \tag{9}$$

Using binomial expansion on the last term in (9) gives:

$$\left(1-e^{-\left[\frac{1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}}{1-\left(1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)}\right]^{2}}\right)^{\theta-1}=\sum_{k=0}^{\infty}(-1)^{k}\binom{\theta-1}{k}\mathrm{e}^{-k\left[\frac{1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}}{1-\left(1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)}\right]^{2}} \tag{10}$$

Substituting the above result into the PDF and carrying out the necessary simplifications yields the following expression:

$$f(x)=\sum_{k=0}^{\infty}\frac{(-1)^{k}\,2\theta\alpha\lambda\beta^{\alpha}x^{\lambda-1}\left(\beta+x^{\lambda}\right)^{-(\alpha+1)}\left(1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)}{\left[1-\left(1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)\right]^{3}}\binom{\theta-1}{k}\mathrm{e}^{-(k+1)\left[\frac{1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}}{1-\left(1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)}\right]^{2}} \tag{11}$$

Recall that base on Taylor series expansion, $\exp(-z)=\sum_{i=0}^{\infty}\frac{(-1)^{i}z^{i}}{i!}$, this implies that the exponential term in PDF of the BPLD in (11) can be expressed as:

$$\mathrm{e}^{-(k+1)\left[\frac{1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}}{1-\left(1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)}\right]^{2}}=\sum_{l=0}^{\infty}\frac{(-1)^{l}(k+1)^{l}}{l!}\left[\frac{1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}}{1-\left(1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)}\right]^{2l} \tag{12}$$

Also, by substituting the above result and simplifying the expression, the PDF in Equation (12) can be written as:

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$$f(x)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{\theta-1}{k}\frac{(-1)^{k+l}(k+1)^{l}}{l!}\frac{2\theta\alpha\lambda\beta^{\alpha}x^{\lambda-1}\left(\beta+x^{\lambda}\right)^{-(\alpha+1)}\left(1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)^{2l+1}}{\left[\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right]^{2l+3}}$$

(13)

Also, using binomial expansion we obtain:

$$\left(1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)^{2l+1}=\sum_{m=0}^{\infty}(-1)^{m}\binom{2l+1}{m}\left(\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)^{m} \quad (14)$$

Again by substituting the result above and simplifying, the pdf in (13) becomes:

$$f(x)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{\theta-1}{k}\binom{2l+1}{m}\frac{(-1)^{k+l+m}}{l!(k+1)^{-l}}\frac{2\theta\alpha\lambda\beta^{\alpha}x^{\lambda-1}}{\left(\beta+x^{\lambda}\right)^{(\alpha+1)}}\left(\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)^{m-2l-3}$$

$$f(x)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{\theta-1}{k}\binom{2l+1}{m}\frac{(-1)^{k+l+m}}{l!(k+1)^{-l}}\frac{2\theta\alpha\lambda x^{\lambda-1}}{\left(\beta^{\alpha}\right)^{-m+2l+2}}\left(x^{\lambda}+\beta\right)^{\alpha(2l-m+2)-1} \quad (15)$$

Let

$$\eta_{klm}=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{\theta-1}{k}\binom{2l+1}{m}\frac{2\theta(-1)^{k+l+m}}{l!(k+1)^{-l}\left(\beta^{\alpha}\right)^{-m+2l+2}}$$

such that the simplified form of the pdf of the BPLD can be expressed as:

$$f(x)=\eta_{klm}\alpha\lambda x^{\lambda-1}\left(x^{\lambda}+\beta\right)^{\alpha(2l-m+2)-1} \quad (16)$$

Hence, Equation (16) provides a simplified linear representation of the PDF, which will be employed in deriving several properties of the BPLD.

## 4.0 STATISTICAL PROPERTIES OF THE BURR X-POWER LOMAX DISTRIBUTION (BPLD)

This section outlines several statistical properties of the BPLD, which are presented as follows:

### 4.1 Moments

Let X denote a continuous random variable the $n^{th}$ moment of X is given by;

$$\mu_n'=E\left(X^n\right)=\int_0^{\infty}x^n f(x)\,dx \quad (17)$$

where f(x), the PDF of the BPLD is as given in Equation (16) as:

$$f(x)=\eta_{klm}\alpha\lambda x^{\lambda-1}\left(x^{\lambda}+\beta\right)^{\alpha(2l-m+2)-1} \quad (18)$$

Substituting Equation (18) in (17) gives;

$$\mu_n'=E\left(X^n\right)=\int_0^{\infty}x^n\left(\eta_{klm}\alpha\lambda x^{\lambda-1}\left(x^{\lambda}+\beta\right)^{\alpha(2l-m+2)-1}\right)dx$$

$$\mu_n'=E\left(X^n\right)=\eta_{klm}\alpha\lambda\int_0^{\infty}x^{n-\lambda-1}\left(\beta+x^{\lambda}\right)^{\alpha(2l-m+2)-1}dx \quad (19)$$

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The following result is obtained by applying the integration by substitution method to Equation (19).

Let $y=\beta+x^{\lambda}\Rightarrow x=(y-\beta)^{\frac{1}{\lambda}}$ such that $\frac{dy}{dx}=\lambda x^{\lambda-1}\Rightarrow dx=\frac{dy}{\lambda x^{\lambda-1}}$.

Substituting for $x$, $y$ and $dx$ in Equation (19) and simplifying yields:

$$\mu_n^{'}=E(X^n)=\eta_{klm}\alpha\lambda\int_0^{\infty}(y-\beta)^{n-\lambda-1}y^{\alpha(2l-m+2)-1}\frac{dy}{\lambda x^{\lambda-1}}$$

$$\mu_n^{'}=E(X^n)=\eta_{klm}\alpha\lambda\int_0^{\infty}(y-\beta)^{\frac{1}{\lambda}(n-\lambda-1)}y^{\alpha(2l-m+2)-1}\frac{dy}{\lambda x^{\lambda-1}}$$

$$\mu_n^{'}=E(X^n)=\eta_{klm}\alpha\int_0^{\infty}(y-\beta)^{\frac{1}{\lambda}(n-\lambda-1)-\frac{1}{\lambda}(\lambda-1)}y^{\alpha(2l-m+2)-1}dy$$

$$\mu_n^{'}=E(X^n)=\eta_{klm}\alpha\int_0^{\infty}(y-\beta)^{\frac{n}{\lambda}}y^{\alpha(2l-m+2)-1}dy$$

$$\mu_n^{'}=E(X^n)=\eta_{klm}\alpha\int_0^{\infty}(-\beta(1-y))^{\frac{n}{\lambda}}y^{\alpha(2l-m+2)-1}dy \tag{20}$$

Rearranging and simplifying Equation (20) produces the following results;

$$\mu_n^{'}=E(X^n)=\eta_{klm}\alpha(-\beta)^{\frac{n}{\lambda}}\int_0^{\infty}y^{\alpha(2l-m+2)-1}(1-y)^{\frac{n}{\lambda}}dy$$

$$\mu_n^{'}=E(X^n)=\eta_{klm}\alpha(-\beta)^{\frac{n}{\lambda}}\int_0^{\infty}y^{\alpha(2l-m+2)-1}(1-y)^{\frac{n}{\lambda}+1-1}dy \tag{21}$$

Recall that $B(x,y)=\int_0^{\infty}t^{x-1}(1-t)^{y-1}\,dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Using the statement above, the $n^{th}$ ordinary moment of X for the BPLD is obtained as:

$$\mu_n^{'}=E(X^n)=\frac{\eta_{klm}\alpha\Gamma(2\alpha l-\alpha m+2\alpha)\Gamma(\frac{n}{\lambda}+1)}{(-\beta)^{-\frac{n}{\lambda}}\Gamma(2\alpha l-\alpha m+2\alpha+\frac{n}{\lambda}+1)} \tag{22}$$

Equation (22) is the $n^{th}$ ordinary moment of X for the BPLD, where

$$\eta_{klm}=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{\theta-1}{k}\binom{2l+1}{m}\frac{2\theta(-1)^{k+l+m}}{l!(k+1)^{-l}(\beta^{\alpha})^{-m+2l+2}}.$$

### 4.2 Moment generating function

The moment generating function of a continuous random variable **X** is given by:

$$M_x(t)=E\left[e^{tx}\right]=\int_{-\infty}^{\infty}e^{tx}f(x)dx \tag{23}$$

Recall that by power series expansion,

$$e^{tx}=\sum_{n=0}^{\infty}\frac{(tx)^n}{n!}=\sum_{n=0}^{\infty}\frac{(t)^n}{n!}x^n \tag{24}$$

Therefore, the moment generating function can also be expressed as:

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$$M_x(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_0^{\infty}x^n f(x)\,dx=\sum_{n=0}^{\infty}\frac{t^n}{n!}E\left(X^n\right)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\left[\mu_n'\right] \tag{25}$$

Using the result in Equation (22) and simplifying yields:

$$M_x(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\left[\frac{\eta_{klm}\alpha\Gamma\left(2\alpha l-\alpha m+2\alpha\right)\Gamma\left(\frac{n}{\lambda}+1\right)}{\left(-\beta\right)^{-\frac{n}{\lambda}}\Gamma\left(2\alpha l-\alpha m+2\alpha+\frac{n}{\lambda}+1\right)}\right] \tag{26}$$

### 4.3 Characteristics function

A representation for the characteristics function of a continuous random variable is given by

$$\phi_x(t)=E\left(e^{itx}\right)=\int_0^{\infty}e^{itx}f(x)\,dx \tag{27}$$

Recall that by power series expansion,

$$\mathrm{e}^{itx}=\sum_{n=0}^{\infty}\frac{(itx)^n}{n!}=\sum_{n=0}^{\infty}\frac{(it)^n}{n!}x^n \tag{28}$$

Hence, simple algebra and use of Equation (28) produces the following results:

$$\phi_x(t)=E\left(\mathrm{e}^{itx}\right)=\sum_{n=0}^{\infty}\frac{(it)^n}{n!}\int_0^{\infty}x^n f(x)\,dx=\sum_{n=0}^{\infty}\frac{(it)^n}{n!}E\left(X^n\right)=\sum_{n=0}^{\infty}\frac{(it)^n}{n!}\left[\mu_n'\right]$$

$$\phi_x(t)=E\left(e^{itx}\right)=\sum_{n=0}^{\infty}\frac{(it)^n}{n!}\left[\frac{\eta_{klm}\alpha\Gamma\left(2\alpha l-\alpha m+2\alpha\right)\Gamma\left(\frac{n}{\lambda}+1\right)}{\left(-\beta\right)^{-\frac{n}{\lambda}}\Gamma\left(2\alpha l-\alpha m+2\alpha+\frac{n}{\lambda}+1\right)}\right] \tag{29}$$

### 4.4 Quantile function, median and simulation

According to Hyndman and Fan (1996), the quantile function for any distribution is defined in the form $Q(u)=F^{-1}(u)$ where $Q(u)$ is the quantile function of $F(x)$ for $0<u<1$

Based on the above definition, the cumulative distribution function of the BPLD is inverted to obtain its corresponding quantile function, as presented below:

$$F(x)=\left\{1-e^{-\left[\frac{1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}}{1-\left[1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right]}\right]^2}\right\}^{\theta}=u \tag{30}$$

Simplifying Equation (30) above gives:

$$Q(u)=\left\{\beta\left(1+\left[-\ln\left(1-u^{\frac{1}{\theta}}\right)\right]^{\frac{1}{2}}\right)^{\frac{1}{\alpha}}-\beta\right\}^{\frac{1}{\lambda}} \tag{31}$$

Using Equation (31), the median of $X$ from the BPLD is simply obtained by setting $u=0.5$ as follows:

$$Median=\left\{\beta\left(1+\left[-\ln\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\theta}}\right)\right]^{\frac{1}{2}}\right)^{\frac{1}{\alpha}}-\beta\right\}^{\frac{1}{\lambda}} \tag{32}$$

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Similarly, random numbers can be simulated from the BPLD by setting $Q(u) = X$ as follows:

$$X = \left\{ \beta \left( 1 + \left[ -\ln\left(1 - u^{\frac{1}{\theta}}\right) \right]^{\frac{1}{2}} \right)^{\frac{1}{\alpha}} - \beta \right\}^{\frac{1}{\lambda}} \tag{33}$$

Moreover, the quartile-based measure of skewness proposed by Kennedy and Keeping is defined as follows:

$$SK = \frac{Q\left(3/4\right) - 2Q\left(1/2\right) + Q\left(1/4\right)}{Q\left(3/4\right) - Q\left(1/4\right)} \tag{34}$$

Relatedly, Moors (1988) presented the Moors' kurtosis based on octiles as:

$$KT = \frac{Q\left(3/8\right) - Q\left(1/8\right) + Q\left(7/8\right) - Q\left(5/8\right)}{Q\left(6/8\right) - Q\left(2/8\right)} \tag{35}$$

"where $Q(.)$ is calculated by using the quantile function from Equation (31).

### 4.5 Reliability Analysis

#### 4.5.1 Survival Function

The probability that a system or an individual survives beyond a specified time is characterized by the survival function (SF), which is mathematically given by:

$$S(x) = 1 - F(x) \tag{36}$$

Now, taking $F(x)$ to be the cdf of the proposed BPLD and substituting, we have;

$$S(x) = 1 - \left\{ 1 - \exp\left\{ -\left[ \frac{1 - \beta^{\alpha}\left(x^{\lambda} + \beta\right)^{-\alpha}}{1 - \left[1 - \beta^{\alpha}\left(x^{\lambda} + \beta\right)^{-\alpha}\right]} \right]^2 \right\} \right\}^{\theta} \tag{37}$$

#### 4.5.2 Hazard Function

The hazard function (HF) measures the risk of failure at a particular instant, conditional on survival up to that time. Mathematically, it is expressed as:

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{f(x)}{S(x)} \tag{38}$$

Substituting for $f(x)$ and $F(x)$, and simplifying gives the hazard function of BPLD as follows:

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$$h(x)=\frac{2\theta\alpha\lambda\beta^{\alpha}x^{\lambda-1}\left(\beta+x^{\lambda}\right)^{-(\alpha+1)}\left[\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right]^{-3}\left(1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)}{\left(1-\left\{1-\exp\left\{-\left[\frac{1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}}{1-\left[1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right]}\right]^{2}\right\}\right\}^{\theta}\right)e^{\left[\frac{1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}}{1-\left(1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)}\right]^{2}}}\left(1-e^{-\left[\frac{1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}}{1-\left(1-\beta^{\alpha}\left(x^{\lambda}+\beta\right)^{-\alpha}\right)}\right]^{2}}\right)^{\theta-1}$$

(39)

Below is a graphical representation of the survival and hazard functions at chosen parameter values. The plots are as given in the figure **2** below:

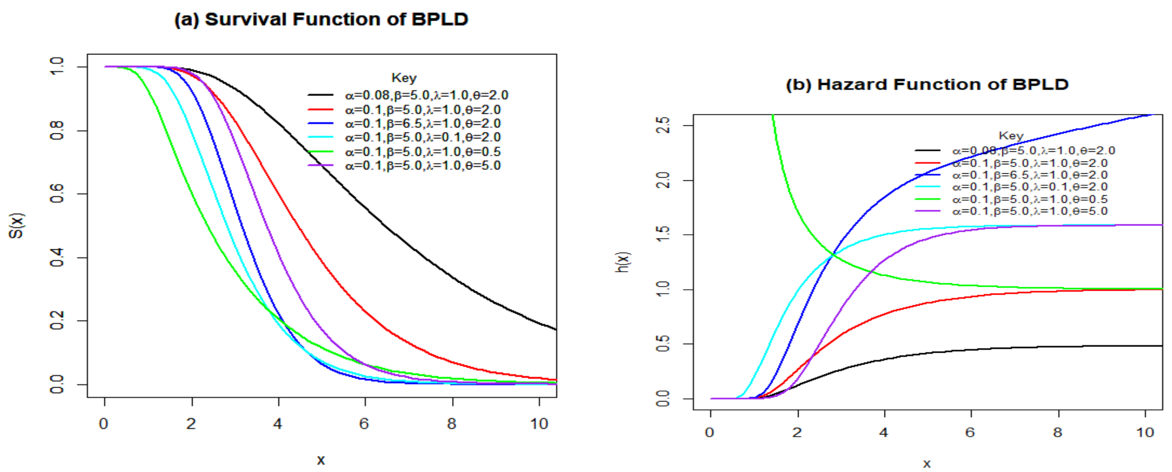


**Figure 2**: The survival and Hazard functions of the BPLD.

Figure 2(a) illustrates that the survival probability of a random variable following the BPLD decreases over time. In other words, as the age of the random variable increases, its probability of survival declines. This behavior indicates that the BPLD is appropriate for modeling lifetime data in which the survival probability diminishes with increasing age. Similarly, Figure 2(b) shows that the hazard (failure) probability of a random variable following the BPLD increases with time. That is, as the age of the random variable advances, the likelihood of failure or death becomes greater. This demonstrates that the BPLD is well suited for modeling lifetime phenomena characterized by an increasing failure rate over time.

## 5.0 MAXIMUM LIKELIHOOD ESTIMATION OF THE UNKNOWN PARAMETERS OF THE BPLD

Let $x_1, x_2, x_3 .........x_n$ be a random sample of size n, consisting of independent and identically distributed observations from the BPLD with unknown parameters α, β, λ and θ as defined earlier.

Using the probability density function given in Equation (8), the likelihood function of the BPLD can be expressed as follows:

$$L\left(\underline{X}\mid\alpha,\beta,\lambda,\theta\right)=\prod_{i=1}^{n}\left\{\frac{2\theta\alpha\lambda\beta^{-2\alpha}x_i^{\lambda-1}\left(\beta+x_i^{\lambda}\right)^{2\alpha-1}}{\left(1-\beta^{\alpha}\left(x_i^{\lambda}+\beta\right)^{-\alpha}\right)^{-1}e^{\left[\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}-1\right]^{2}}}\left(1-e^{-\left[\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}-1\right]^{2}}\right)^{\theta-1}\right\} \quad (40)$$

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Let the natural logarithm of the likelihood function be, $l = \log L\left(\underline{X} \mid \alpha, \beta, \lambda, \theta\right)$, therefore, taking the natural logarithm of the function Equation (40) above gives:

$$l = n\log(2\alpha) + n\log\lambda + n\log\theta - 2\alpha n\log\beta + (\lambda-1)\sum_{i=1}^{n}\log x_i + (2\alpha-1)\sum_{i=1}^{n}\log\left(\beta + x_i^{\lambda}\right) + \sum_{i=1}^{n}\log\left[1-\beta^{\alpha}\left(x_i^{\lambda}+\beta\right)^{-\alpha}\right]$$
$$-\left[\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}-1\right]^2 + (\theta-1)\sum_{i=1}^{n}\log\left(1-e^{-\left[\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}-1\right]^2}\right)$$

(41)

Differentiating $l$ partially with respect to $\alpha$, $\beta$, $\lambda$, and $\theta$ respectively gives the following results:

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} - 2n\log\beta + 2\sum_{i=1}^{n}\log\left(\beta + x_i^{\lambda}\right) + \sum_{i=1}^{n}\left\{\frac{\beta^{\alpha}\left(x_i^{\lambda}+\beta\right)^{-\alpha}\left[\ln\left(x_i^{\lambda}+\beta\right)-\ln\beta\right]}{\left[1-\beta^{\alpha}\left(x_i^{\lambda}+\beta\right)^{-\alpha}\right]}\right\}$$
$$+2\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}\left[\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}-1\right]\left[\ln\beta - \ln\left(x_i^{\lambda}+\beta\right)\right]$$
$$-(\theta-1)\sum_{i=1}^{n}\left[\frac{2\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}\left[\ln\beta-\ln\left(x_i^{\lambda}+\beta\right)\right]e^{-\left[\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}-1\right]^2}}{\left[\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}-1\right]^{-1}\left(1-e^{-\left[\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}-1\right]^2}\right)}\right] \quad (42)$$

$$\frac{\partial l}{\partial\beta} = -\frac{2\alpha n}{\beta} - (2\alpha-1)\sum_{i=1}^{n}\left\{\frac{1}{\left(\beta + x_i^{\lambda}\right)}\right\} - \sum_{i=1}^{n}\left\{\frac{\alpha\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}\left[\beta^{-1}+\left(x_i^{\lambda}+\beta\right)^{-1}\right]}{\left[1-\beta^{\alpha}\left(x_i^{\lambda}+\beta\right)^{-\alpha}\right]}\right\}$$
$$+2\alpha\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}\left[\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}-1\right]\left[\beta^{-1}+\left(x_i^{\lambda}+\beta\right)^{-1}\right]$$
$$-(\theta-1)\sum_{i=1}^{n}\left[\frac{2\alpha\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}\left[\beta^{-1}+\left(x_i^{\lambda}+\beta\right)^{-1}\right]e^{-\left[\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}-1\right]^2}}{\left[\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}-1\right]^{-1}\left(1-e^{-\left[\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}-1\right]^2}\right)}\right]$$

(43)

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$$\frac{\partial l}{\partial \lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\log x_i+(2\alpha-1)\sum_{i=1}^{n}\left\{\frac{x_i^{\lambda}\ln x_i}{\left(\beta+x_i^{\lambda}\right)}\right\}-\sum_{i=1}^{n}\left\{\frac{\alpha\beta^{\alpha}x_i^{\lambda}\ln x_i\left(x_i^{\lambda}+\beta\right)^{-\alpha-1}}{\left[1-\beta^{\alpha}\left(x_i^{\lambda}+\beta\right)^{-\alpha}\right]}\right\}$$

$$-2\alpha\beta^{-\alpha}x_i^{\lambda}\ln x_i\left(x_i^{\lambda}+\beta\right)^{\alpha-1}\left[\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}-1\right]$$

$$+(\theta-1)\sum_{i=1}^{n}\left[\frac{2\alpha\beta^{-\alpha}x_i^{\lambda}\ln x_i\left(x_i^{\lambda}+\beta\right)^{\alpha-1}e^{-\left[\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}-1\right]^2}}{\left[\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}-1\right]^{-1}\left(1-e^{-\left[\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}-1\right]^2}\right)}\right] \quad (44)$$

$$\frac{\partial l}{\partial \theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\log\left(1-e^{-\left[\beta^{-\alpha}\left(x_i^{\lambda}+\beta\right)^{\alpha}-1\right]^2}\right) \quad (45)$$

Setting equations (42), (43), (44), and (45) equal to zero and solving the resulting system of nonlinear equations yields the maximum likelihood estimates of the parameters, $\alpha$, $\beta$, $\lambda$ and $\theta$

However, obtaining closed-form analytical solutions is not straightforward. As a result, numerical techniques—particularly the Newton–Raphson iterative method—are employed, typically implemented using computational tools such as Python, R, MATLAB, or similar software.

## 6.0 SUMMARY AND CONCLUSION

This study introduced a new generalization of the Lomax distribution, referred to as the Burr X-Power Lomax Distribution (BPLD). Several statistical properties of the proposed model were derived and examined, including the quantile function, ordinary moments, moment generating function, characteristic function, survival function, and hazard function. The unknown parameters of the distribution were estimated using the maximum likelihood estimation (MLE) technique. Furthermore, a Monte Carlo simulation study was carried out to evaluate the performance of the parameter estimators under different sample sizes.

The practical applicability of the proposed distribution was demonstrated using three real-life datasets. The probability density and hazard function plots revealed that the BPLD is highly flexible, with its shape varying according to the parameter values. Depending on these values, the distribution exhibited several forms, including decreasing, unimodal, right-skewed, nearly symmetric, and asymmetric patterns. The survival function indicated that the BPLD is appropriate for modelling lifetime or age-related phenomena in which the probability of survival declines over time. In addition, the cumulative distribution and survival function plots confirmed that the BPLD satisfies the properties of a valid probability distribution.

The findings from the Monte Carlo simulation further showed that, as the sample size increased, the estimated parameters converged toward their true values, while both the estimation bias and mean squared error decreased. These results are consistent with the fundamental principles of statistical estimation, demonstrating the consistency and improved accuracy of the maximum likelihood estimators for the proposed distribution.

## REFERENCES

Agbor, B. U., Abbas, U. F., Hamma, S., Makmun, F. K., Jah, P. D., & Mark, T. I. (2022). Modelling mother-to-child HIV transmission rate in Nigeria using a Lindley-Lomax distribution. Asian Journal of Probability and Statistics, 20(4), 184–199. https://doi.org/10.9734/AJPAS/2022/v20i4448

Al-Babtain, A. A., Elbatal, I., Al-Mofleh, H., Gemeay, A. M., Afify, A. Z., & Sarg, A. M. (2021). The flexible Burr X-G family: Properties, inference, and applications in engineering science. Symmetry, 13(3), 474. https://doi.org/10.3390/sym13030474

Alotaibi, N., Elbatal, I., Almetwally, E. M., Alyami, S. A., Al-Moisheer, A. S., & Elgarhy, M. (2022). Truncated Cauchy power Weibull-G class of distributions: Bayesian and non-Bayesian inference modelling for COVID-19 and carbon fiber data. Mathematics, 10(9), 1565. https://doi.org/10.3390/math10091565

Al-Shomrani, A. A., & Al-Arfaj, A. (2021). A new flexible odd Kappa-G family of distributions: Theory and properties. Advances and Applications in Mathematical Sciences, 21, 3443–3514.

Anzagra, L., Sarpong, S., & Nasiru, S. (2022). Odd Chen-G family of distributions. Annals of Data Science, 9(2), 369–391. https://doi.org/10.1007/s40745-020-00248-2

Bantan, R. A., Chesneau, C., Jamal, F., Elbatal, I., & Elgarhy, M. (2021). The truncated Burr X-G family of distributions: Properties and applications to actuarial and financial data. Entropy, 23(9), 1088. https://doi.org/10.3390/e23091088

Benchiha, S., Sapkota, L. P., Al Mutairi, A., Kumar, V., Khashab, R. H., Gemeay, A. M., Elgarhy, M., & Nassr, S. G. (2023). A new sine family of generalized distributions: Statistical inference with applications. Mathematics and Computers in Application, 28(5), 83. https://doi.org/10.3390/mca28050083

Chipepa, F., Moakofi, T., & Oluyede, B. (2022). The Marshall-Olkin-odd power generalized Weibull-G family of distributions with applications to COVID-19 data. Journal of Probability and Statistical Science, 20, 1–20.

Eghwerido, J. T., & Agu, F. I. (2021). The shifted Gompertz-G family of distributions: Properties and applications. Mathematica Slovaca, 71(5), 1291–1308.

Eghwerido, J. T., Agu, F. I., & Ibidoja, O. J. (2022). The shifted exponential-G family of distributions: Properties and applications. Journal of Statistics and Management Systems, 25(1), 43–75.

Elbatal, I., Alotaibi, N., Almetwally, E. M., Alyami, S. A., & Elgarhy, M. (2022). On odd Perks-G class of distributions: Properties, regression model, discretization, Bayesian and non-Bayesian estimation, and applications. Symmetry, 14(5), 883. https://doi.org/10.3390/sym14050883

Fayomi, A., Almetwally, E. M., & Qura, M. E. (2023). A novel bivariate Lomax-G family of distributions: Properties, inference, and applications to environmental, medical, and computer science data. AIMS Mathematics, 8, 17539–17584.

Gabanakgosi, M., & Oluyede, B. (2023). The Topp-Leone type II exponentiated half logistic-G family of distributions with applications. International Journal of Mathematics in Operational Research, 25, 85–117.

Hyndman, R. J., & Fan, Y. (1996). Sample quantiles in statistical packages. The American Statistician, 50(4), 361–365.

Ieren, T. G., & Balogun, O. S. (2021). Exponential-Lindley distribution: Theory and application to bladder cancer data. Journal of Applied Probability and Statistics, 16(2), 129–146.

Ieren, T. G., Abdulkadir, S. S., Okolo, A., & Jibasen, D. (2024). A new Fréchet-G family of continuous probability distributions: Special models, properties, simulation and applications. Journal of the Royal Statistical Society Nigeria Group, 1(1), 46–71.

Isa, N. M., Muhammad, M. D., Yahaya, A., & Yakubu, A. (2022a). Sine-exponential distribution: Properties and applications. FUDMA Journal of Sciences, 6(4), 180–194.

Isa, N. M., Muhammad, M. D., Yahaya, A., & Yakubu, A. (2022b). Sine Burr XII distribution: Statistical properties and applications. FUDMA Journal of Sciences, 6(4), 195–212.

Joel, J., Ieren, T. G., Yakura, B. S., Aniah-Betiang, E. I., & Iseyemi, S. O. (2024). A sine Lomax-exponential distribution: Its properties, simulation and applications to survival data. African Journal of Mathematics and Statistics Studies, 7(4), 296–319. https://doi.org/10.52589/AJMSS-IHSYZU29

Kenney, J. F., & Keeping, E. S. (1962). Mathematics of statistics (3rd ed.). Chapman & Hall.

Lomax, K. S. (1954). Business failures: Another example of the analysis of failure data. Journal of the American Statistical Association, 49(268), 847–852.

Méndez-González, L. C., Rodríguez-Picón, L. A., Rodríguez Borbón, M. I., & Hansuk, S. (2023). The Chen–Perks distribution: Properties and reliability applications. Mathematics, 11(13), 3001. https://doi.org/10.3390/math11133001

Mohammad, S. (2024). X-exponential-G family of distributions with applications. International Journal of Statistics and Probability, 13(1), 40–54. https://doi.org/10.5539/ijsp.v13n1p40

Moors, J. J. A. (1988). A quantile alternative for kurtosis. Journal of the Royal Statistical Society: Series D (The Statistician), 37(1), 25–32.

Mustapha, B. A., Isa, A. M., Sule, O. B., & Itopa, I. I. (2023). Sine-Lomax distribution: Properties and applications to real data sets. FUDMA Journal of Sciences, 7(4). https://doi.org/10.33003/fjs-2023-0704-1904

Umar, S. A., Bukar, A. B., Makama, M. S., & Ieren, T. G. (2021). Some results on the transmuted odd Lindley-Rayleigh distribution. Benin Journal of Statistics, 4, 135–153.