

# A Multi-Intervention Framework for Diphtheria Eradication: Optimizing Public Health Via Vaccination, Early Diagnosis, Isolation and Community-Driven Infection Control

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## ABSTRACT

This study develops and analyzes a mathematical model for diphtheria transmission incorporating vaccination, exposure prevention, and treatment interventions. Analytical results include the disease-free and endemic equilibria, the basic reproduction number ( $R_0$ ), and a sensitivity analysis showing that transmission rate, vaccination coverage, and treatment effort have the greatest influence on disease spread. A Hopf bifurcation analysis identifies threshold parameter values where the system may transition to oscillatory behavior, highlighting the potential for periodic outbreaks under specific conditions. Using the Pontryagin Maximum Principle, an optimal control framework is implemented via the forward-backward sweep method to minimize the number of infected individuals while optimizing public health resource allocation. Numerical simulations demonstrate that combined interventions significantly reduce both asymptomatic and symptomatic infections, achieving disease elimination within approximately 60 days. These results provide quantitative guidance for public health strategies, emphasizing the importance of early, sustained, and integrated control measures for effective diphtheria management

**Keywords:** Diphtheria transmission dynamics, basic reproduction number, sensitivity analysis, Hopf bifurcation and optimal control theory

## 1.0 INTRODUCTION

Diphtheria is a highly contagious, toxin-mediated bacterial infection caused primarily by *Corynebacterium diphtheriae*, and less frequently by *Corynebacterium ulcerans* and *Corynebacterium pseudotuberculosis*. The disease is transmitted mainly through respiratory droplets or close physical contact with infected individuals and is characterized clinically by sore throat, fever, cervical lymphadenopathy, and the formation of a grayish pseudomembrane in the upper respiratory tract, which can lead to airway obstruction and death if untreated (Abbas *et al.*, 2025; World Health Organization [WHO], 2017).

Despite the availability of an effective vaccine for over seven decades, diphtheria remains a public health concern in many low- and middle-income countries, particularly where routine immunization coverage is suboptimal or disrupted by conflict, population displacement, and weak health systems (Clarke, 2019; Erim *et al.*, 2025). Periodic outbreaks continue to occur in parts of Africa and Asia, underscoring the persistence of immunity gaps and the challenges of sustaining herd immunity (WHO, 2023; WHO, 2024; Denué *et al.*, 2024). Inadequate booster vaccination in adolescents and adults further contributes to disease resurgence, as immunity wanes over time (Truelove *et al.*, 2020; Okhuarobo *et al.*, 2025).

From a public health perspective, diphtheria control relies on a combination of preventive vaccination, early case detection, prompt administration of diphtheria antitoxin, antibiotic therapy, and effective isolation of infectious individuals (Taccari, Smith, & Lopez, 2023). However, logistical constraints, delayed diagnosis, and limited access to antitoxin in resource-limited settings often undermine these control efforts. These challenges highlight the need

for robust analytical tools that can support evidence-based decision-making and optimize intervention strategies.

Mathematical modeling has emerged as a powerful framework for understanding the transmission dynamics of infectious diseases such as diphtheria. Compartmental models enable researchers to quantify the impact of key epidemiological factors, including vaccination, waning immunity, recovery, and disease-induced mortality, on disease persistence and control (Truelove *et al.*, 2020; Yusefi, 2025). Furthermore, the integration of stability analysis, bifurcation theory, and numerical simulations provides insight into threshold conditions for disease elimination and the potential for complex dynamics such as oscillatory outbreaks.

In recent years, optimal control theory has been increasingly applied to infectious disease models to identify cost-effective strategies for minimizing disease burden while accounting for limited resources (Lenhart & Workman, 2007; Prates, Silva, & Albuquerque, 2025). For diphtheria, such approaches are particularly relevant in designing vaccination and treatment policies that balance public health impact with economic feasibility. Consequently, a rigorous analytical and numerical investigation of diphtheria transmission dynamics is essential for informing sustainable control and eradication strategies, especially in endemic and outbreak-prone regions.

## 2.0 MATHEMATICAL MODEL

### 2.1 Model Assumptions

The population is assumed to mix homogeneously, meaning every individual has an equal probability of coming into effective contact with an infectious individual.

Individuals enter the population at a constant recruitment rate  $\Lambda$  and are subject to natural death at a constant rate  $\mu$ . Disease-induced mortality from diphtheria is assumed negligible due to timely medical intervention.

Vaccination provides effective protection against diphtheria. Vaccinated individuals move to the vaccinated compartment and do not contribute to transmission. Waning immunity is neglected over the time horizon of the study.

After infection, individuals pass through an exposed (latent) stage during which they are infected but not infectious. Progression from exposure to infectious classes occurs at a constant rate.

Infectious individuals are categorized as asymptomatic ( $I_A$ ) or symptomatic ( $I_S$ ), reflecting different clinical presentations and transmission potentials.

Diphtheria is transmitted only through person-to-person contact; vertical transmission (from mother to child) is not considered.

The model diagram illustrates the transmission dynamics of diphtheria in a human population, which is divided into seven epidemiological compartments: Susceptible ( $S$ ), Vaccinated ( $V$ ), Exposed ( $E$ ), Quarantined ( $Q$ ), Asymptomatic Infectious ( $I_A$ ), Symptomatic Infectious ( $I_S$ ), and Recovered ( $R$ ). New individuals enter the population through recruitment at a constant rate ( $\Lambda$ ) and are assumed to join the susceptible class ( $S$ ). Susceptible individuals may acquire diphtheria through effective contact with infectious individuals ( $I_A$ ) and ( $I_S$ ), moving into the exposed class ( $E$ ) at a rate governed by the force of infection ( $\lambda$ ). Vaccination is administered to susceptible individuals at rate ( $\gamma$ ), moving them into the vaccinated class ( $V$ ). This process is enhanced by the vaccination control ( $u_1$ ), which reduces susceptibility to infection.

## 2.2 Description of the Model Diagram

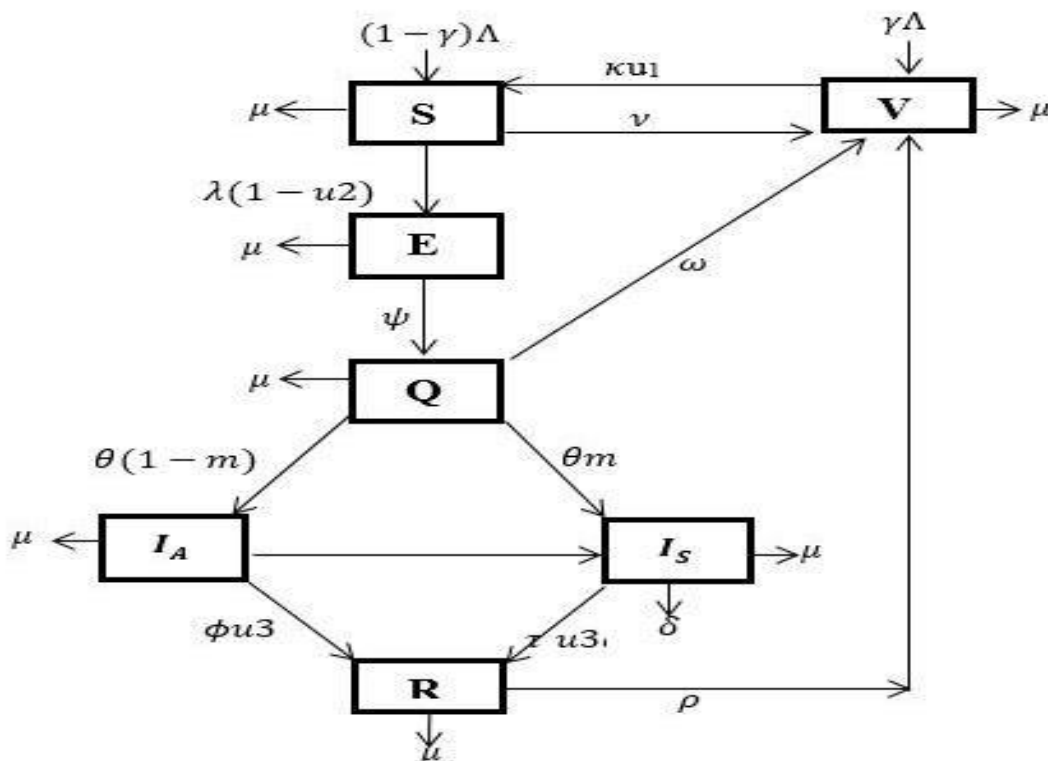


Figure 1: Schematic diagram of Diphtheria (Hycienth, et al., 2025)

Exposed individuals progress to infection after the incubation period. A proportion  $m$  of exposed individuals is identified early and transferred to the quarantined class ( $Q$ ), while the remaining proportion  $(1-m)$  develops asymptomatic infection ( $I_A$ ). Quarantined individuals may progress to become symptomatically infectious ( $I_S$ ), reflecting clinical

progression despite isolation. Asymptomatic infectious individuals ( $I_A$ ) may recover naturally or through treatment and move into the recovered class ( $R$ ). Symptomatic infectious individuals ( $I_S$ ) also recover through treatment at a higher rate due to medical intervention. The treatment control ( $u_3$ ) enhances recovery from both infectious classes, thereby reducing the



infectious period and limiting further transmission. The exposure reduction control ( $u_2$ ) acts on the force of infection by reducing effective contact between susceptible and infectious individuals. This control captures public health measures such as isolation, hygiene practices, and social distancing. Vaccinated and recovered individuals are

assumed to have immunity, and all compartments are subject to natural death at rate ( $\mu$ ). The combined effect of vaccination, quarantine, treatment, and exposure reduction determines the overall disease dynamics and potential for diphtheria elimination.

Table 1. Model Variables/Parameters Description

Symbol	Variables/parameters	Description
S(t)	Susceptible population	Individuals who are not infected with diphtheria and have not been vaccinated
V(t)	Vaccinated population	Individuals vaccinated against diphtheria and protected from infection
E(t)	Exposed population	Individuals infected but not yet infectious (latent stage)
Q(t)	Quarantined population	Exposed individuals identified early and isolated
$I_A(t)$	Asymptomatic infectious	Infected individuals without clinical symptoms but capable of transmission
$I_S(t)$	Symptomatic infectious	Infected individuals with symptoms and higher infectiousness
R(t)	Recovered population	Individuals who have recovered and acquired permanent immunity
$\Lambda$	Recruitment rate	Rate at which individuals enter the susceptible population
$\mu$	Natural death rate	Natural mortality rate for all compartments
$\beta$	Transmission rate	Effective contact rate between susceptible and infectious individuals
$\eta_A$	Relative infectiousness	Infectiousness of asymptomatic individuals relative to symptomatic cases
$\gamma$	Vaccination rate	Rate at which susceptible individuals are vaccinated
$\sigma$	Progression rate	Rate at which exposed individuals leave the latent stage
M	Quarantine proportion	Proportion of exposed individuals placed under quarantine
$\rho$	Quarantine progression rate	Rate at which quarantined individuals develop symptomatic infection
$\alpha_A$	Recovery rate (asymptomatic)	Rate of recovery for asymptomatic infectious individuals
$\alpha_S$	Recovery rate (symptomatic)	Rate of recovery for symptomatic infectious individuals
$\delta$	Disease-induced death rate	Mortality rate due to diphtheria (if included)
$u_1(t)$	Vaccination control	Effort applied to increase vaccination coverage
$u_2(t)$	Exposure reduction control	Effort applied to reduce effective contact (e.g., isolation, hygiene)
$u_3(t)$	Treatment control	Effort applied to improve treatment and recovery

Table 2. Model Parameters/ Variables Description and Value Source/Remark

Parameter/Variables	Meaning	Value	Source/Remark
$\Lambda$	Recruitment rate	1958	Assumed
$\mu$	Natural death rate	0.02	(WHO, 2023)
$\beta$	Transmission rate	0.6	Estimated
$\nu$	Vaccination rate	0.4	(Lenhart & Workman, 2007).
$\gamma$	Proportion vaccinated at birth	0.6	Assumed
$\psi$	Progression rate (E to Q)	0.25	(Truelove <i>et al.</i> , 2020)



Parameter/Variables	Meaning	Value	Source/Remark
$\theta$	Progression rate (Q to I)	0.4	Estimated
M	Symptomatic fraction	0.65	(WHO, 2017)
$\phi$	Recovery (asymptomatic)	0.3	Estimated
$\delta$	Disease-induced death	0.05	(WHO, 2023)
$\rho$	Loss of immunity	0.1	(Truelove <i>et al.</i> , 2020)
S	Susceptible population	470	
V	Vaccinated population	1488	
E	Exposed population	0	
Q	Quarantined population	0	
$I_A(t)$	Asymptomatic infectious	0	
$I_S(t)$	Symptomatic infectious	0	
R	Recovered population	0	

### 2.3: The Model Equations

$$\left. \begin{aligned}
 \frac{dV}{dt} &= b\Lambda + vS + \omega Q + \rho R - \kappa u_1 V - \mu V \\
 \frac{dS}{dt} &= \kappa u_1 V + (1 - \gamma)\Lambda - vS - \mu S - \frac{\beta(\sigma I_A + I_S)}{N}(1 - u_2)S \\
 \frac{dE}{dt} &= \frac{\beta(\sigma I_A + I_S)}{N}(1 - u_2)S - \mu E - \psi E \\
 \frac{dQ}{dt} &= \psi E - \omega Q - \mu Q - \theta_m Q \\
 \frac{dI_A}{dt} &= \theta(1 - m)Q - \mu I_A - \alpha I_A - \phi u_3 I_A \\
 \frac{dI_S}{dt} &= \theta_m Q + \alpha I_A - \mu I_S - \delta I_S - \tau u_3 I_S \\
 \frac{dR}{dt} &= \phi u_3 I_A + \tau u_3 I_S - \mu R - \rho R
 \end{aligned} \right\} \tag{1}$$

with the following initial conditions

$$V(0) \geq 0, S(0) \geq 0, E(0) > 0, Q(0) > 0, I_A(0) > 0, I_S(0) > 0, R(0) > 0$$

### 3.0 Model Analysis

In this section, we obtain the following results which guarantee that the diphtheria disease is governed by the system (1) is epidemiological and mathematical meaningful in a region given by

$$S_0 = \left\{ \begin{aligned}
 &\{V, S, E, I_A, I_S, Q, R\} \in \mathbb{R}_+^7, V > S > E \geq 0, I_A \geq 0, \\
 &I_S \geq 0, Q \geq 0, R \geq 0, N_{(t)} \leq \frac{\Lambda}{\mu} + \left( \mu_0 - \frac{\Lambda}{\mu} \right) e^{-\mu t}
 \end{aligned} \right\}$$

3.1 Disease-free and endemic equilibrium point of the diphtheria model is given as

$$\mathbb{Z}_0 = (V_0, S_0, E_0, Q_0, I_{A0}, I_{S0}, R_0) = \left\{ \frac{\gamma\Lambda(v\kappa u_1 + v\mu + \mu^2) + (\Lambda\kappa u_1 + \Lambda\mu - \gamma\Lambda\mu)v}{(v\kappa u_1 + v\mu + \mu^2)(\kappa u_1 + \mu)}, \frac{\Lambda\kappa u_1 + \Lambda\mu - \gamma\Lambda\mu}{v\kappa u_1 + v\mu + \mu^2}, 0, 0, 0, 0, 0 \right\}$$

$$\left. \begin{aligned} V^* &= \frac{(\gamma + v)S^*}{\mu} \\ S^* &= \frac{\Lambda}{\lambda^* + \mu + \gamma + v} \\ E^* &= \frac{\lambda^* S^*}{\mu + \psi} \\ Q^* &= \frac{m\psi}{\mu + \phi} E^* \\ I_A^* &= \frac{(1-m)\psi}{\mu + \delta} E^* \\ I_S^* &= \frac{\theta m\psi}{(\mu + \eta)(\mu + \phi)} E^* \\ R^* &= \frac{I_A^* \delta + \eta I_S^*}{\mu} \end{aligned} \right\} \text{ respectively}$$

and  $\lambda^* = \frac{\beta(\sigma I_A^* + I_S^*)}{N}$

### 3.2 Reproduction Number for Diphtheria

Diphtheria,  $R_0$  represents the expected number of new diphtheria cases generated by one individual infected with *Corynebacterium diphtheriae* through effective respiratory or close-contact transmission when introduced into a wholly susceptible community. This threshold quantity determines whether diphtheria can invade and persist in the

population or will eventually die out. When vaccination, treatment, or other control strategies are present, the relevant threshold parameter becomes the effective or control reproduction number, denoted by  $R_e$  or  $R_c$ , which measures the average number of secondary diphtheria infections produced by an infectious individual in a partially immune population.

From the system (1) we have

$$F = \begin{bmatrix} 0 & 0 & A & B \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2)$$

$$\frac{\beta\sigma}{N}(1-u_2)S = A, \frac{\beta}{N}(1-u_2)S = B$$

$$V = \begin{bmatrix} C & 0 & 0 & 0 \\ -\psi & D & 0 & 0 \\ 0 & -E & F & 0 \\ 0 & -I & -\alpha & G \end{bmatrix} \quad (3)$$

$$\begin{aligned} (\mu + \psi) &= C, (\omega + \mu + \theta_m) = D, \\ -(\theta - \theta_m) &= E, \mu + \alpha + \phi u_3 = F, \\ \mu + \delta + \tau u_3 &= G, \theta_m = I \end{aligned}$$

Taking the inverse of equation (3) we have

$$V^{-1} = \begin{bmatrix} \frac{1}{C} & 0 & 0 & 0 \\ \frac{1}{C} \frac{\psi}{D} & \frac{1}{D} & 0 & 0 \\ \frac{1}{CF} \frac{\psi}{D} E & \frac{1}{DF} E & \frac{1}{F} & 0 \\ \frac{1}{CFGD} (F\psi I + \alpha\psi E) & \frac{1}{FGD} (FI + \alpha E) & \frac{1}{FG} \alpha & \frac{1}{G} \end{bmatrix} \quad (4)$$

Solving the reproduction number using equation (2) and (3) at  $FV^{-1}$  we have

$$\begin{aligned} R_0 &= \frac{1}{CFGD} (AG\psi E + BF\psi I + B\alpha\psi E) \\ R_0 &= \frac{(\beta\sigma(1-u_2)(\mu + \delta + \tau u_3)\psi(\theta - \theta_m) + \beta(1-u_2)(\mu + \alpha + \phi u_3)\psi\theta_m + \beta(1-u_2)\alpha\psi(\theta - \theta_m))}{(\mu + \psi)(\mu + \alpha + \phi u_3)(\mu + \delta + \tau u_3)(\omega + \mu + \theta_m)} \quad (5) \end{aligned}$$

### 3.3 Sensitivity Analysis

Table 3: The table below are the parameters that are contributing to the spread of diphtheria.

Parameter type	Parameters	Sensitivity
Transmission	$\beta$	Yes
Progression	$\psi$	Yes
Quarantine	$\phi$	Yes
Treatment	$\delta$	Yes
Fraction symptomatic	$\theta_m$	Yes
Natural death	$\mu$	Yes
Faster progression increase	$\theta_m$	Yes

### 3.4 Hopf Bifurcation

Hopf bifurcation is used to identify conditions under which a stable endemic equilibrium of the diphtheria model loses stability and gives rise to sustained oscillations in disease prevalence. These Oscillations represent recurrent outbreaks driven by changes in key parameters such as the transmission rate or waning immunity.

The system (1) can be written as

$$\begin{aligned}
 f_1 &= b\Lambda + vS + \omega Q + \rho R - \kappa u_1 V - \mu V \\
 f_2 &= \kappa u_1 V + (1 - \gamma)\Lambda - vS - \mu S - \frac{\beta(\sigma I_A + I_S)}{N}(1 - u_2)S \\
 f_3 &= \frac{\beta(\sigma I_A + I_S)}{N}(1 - u_2)S - \mu E - \psi E \\
 f_4 &= \psi E - \omega Q - \mu Q - \theta_m Q \\
 f_5 &= \theta(1 - m)Q - \mu I_A - \alpha I_A - \phi u_3 I_A \\
 f_6 &= \theta_m Q + \alpha I_A - \mu I_S - \delta I_S - \tau u_3 I_S \\
 f_7 &= \phi u_3 I_A + \tau u_3 I_S - \mu R - \rho R
 \end{aligned} \tag{6}$$

Solving equation (6) at the endemic we have

$$\begin{aligned}
 \frac{df_1}{dV} &= -\kappa u_1 - \mu, \frac{df_1}{dS} = v, \frac{df_1}{dE} = 0, \frac{df_1}{dQ} = \omega, \frac{df_1}{dI_A} = 0, \frac{df_1}{dI_S} = 0, \frac{df_1}{dR} = 0, \\
 \frac{df_2}{dV} &= \kappa u_1, \frac{df_2}{dS} = \frac{\beta(\sigma I_A + I_S)}{N}(1 - u_2), \frac{df_2}{dE} = 0, \frac{df_2}{dQ} = 0, \frac{df_2}{dI_A} = -\frac{\beta(\sigma)}{N}(1 - u_2)S, \frac{df_2}{dI_S} = -\frac{\beta}{N}(1 - u_2)S, \\
 \frac{df_3}{dV} &= 0, \frac{df_3}{dS} = \frac{\beta(\sigma I_A + I_S)}{N}(1 - u_2), \frac{df_3}{dE} = -\mu - \psi, \frac{df_3}{dQ} = 0, \frac{df_3}{dI_A} = \frac{\beta(\sigma)}{N}(1 - u_2)S, \frac{df_3}{dI_S} = -\frac{\beta}{N}(1 - u_2)S, \frac{df_3}{dR} = 0, \\
 \frac{df_4}{dV} &= 0, \frac{df_4}{dS} = 0, \frac{df_4}{dE} = \psi, \frac{df_4}{dQ} = -\omega - \mu - \theta_m, \frac{df_4}{dI_A} = 0, \frac{df_4}{dI_S} = 0, \frac{df_4}{dR} = 0, \\
 \frac{df_5}{dV} &= 0, \frac{df_5}{dS} = 0, \frac{df_5}{dE} = 0, \frac{df_5}{dQ} = \theta(1 - m), \frac{df_5}{dI_A} = -\mu - \alpha - \phi u_3, \frac{df_5}{dI_S} = 0, \frac{df_5}{dR} = 0, \\
 \frac{df_6}{dV} &= 0, \frac{df_6}{dS} = 0, \frac{df_6}{dE} = 0, \frac{df_6}{dQ} = \theta_m, \frac{df_6}{dI_A} = \alpha, \frac{df_6}{dI_S} = -\mu - \delta - \tau u_3, \frac{df_6}{dR} = 0, \\
 \frac{df_7}{dV} &= 0, \frac{df_7}{dS} = 0, \frac{df_7}{dE} = 0, \frac{df_7}{dQ} = 0, \frac{df_7}{dI_A} = \phi u_3, \frac{df_7}{dI_S} = \tau u_3, \frac{df_7}{dR} = -\mu - \rho
 \end{aligned}$$

$$\begin{bmatrix}
 -\kappa u_1 - \mu & v & 0 & \omega & 0 & 0 & \rho \\
 \kappa u_1 & -v - \mu - \frac{\beta(\sigma I_A + I_S)}{N}(1 - u_2) & 0 & 0 & -\frac{\beta(\sigma)}{N}(1 - u_2)S & -\frac{\beta}{N}(1 - u_2)S & 0 \\
 0 & \frac{\beta(\sigma I_A + I_S)}{N}(1 - u_2) & -\mu - \psi & 0 & \frac{\beta(\sigma)}{N}(1 - u_2)S & \frac{\beta}{N}(1 - u_2)S & 0 \\
 0 & 0 & \psi & -\omega - \mu - \theta_m & 0 & 0 & 0 \\
 0 & 0 & 0 & \theta(1 - m) & -\mu - \alpha - \phi u_3 & 0 & 0 \\
 0 & 0 & 0 & \theta_m & \alpha & -\mu - \delta - \tau u_3 & 0 \\
 0 & 0 & 0 & 0 & \phi u_3 & \tau u_3 & -\mu - \rho
 \end{bmatrix} \tag{7}$$

Solving equation (7), we have

$$\beta = \bar{\beta} = \frac{\kappa u_1 + 7\mu + \nu + \psi + \omega + \theta_m + \alpha + \phi u_3 + \delta + \tau u_3 + \rho}{\frac{\beta(\sigma I_A + I_S)}{N}(1-u_2)} \quad (8)$$

The condition for the Hopf-bifurcation is satisfied as  $\beta = \bar{\beta}$ . Now, the second condition is  $\frac{dT}{d\beta} \neq 0$ . From equation (7) finding the trace of the matrix (T) and differentiating same with respect to  $\beta$  we have

$$\frac{dT}{d\beta} = \frac{(\sigma I_A + I_S)}{N}(1-u_2) \neq 0$$

Hence, the system at the parameter  $\beta = \bar{\beta}$  has a Hopf-Bifurcation around the endemic equilibrium which means that the  $I_A(t)$  Asymptomatic infectious and  $I_S(t)$  Symptomatic Infectious have to be controlled or else an endemic will occur (In line with Washachi *et al.*, 2021)

### 3.5 Optimal Control

Optimal control is applied to determine time-dependent vaccination, exposure prevention. And treatment strategies that minimize diphtheria infections while reducing intervention costs. Using pontryagin's maximum principle, the approach identifies optimal policy paths that balance disease reduction and resource use.

From the system (1) of the equation we determine the:

Objective functional

$$J = \int_0^T \left( A_1 I_A + A_2 I_S + \frac{1}{2} (B_1 u_1^2 + B_2 u_2^2 + B_3 u_3^2) \right) dt \quad (9)$$

**Theorem 1.0** Lenhart and Workman (2007) states that if  $u_i^*(t)$  and  $x^*(t)$  are optimal for problem (1), then there exists a piecewise differentiable adjoint variable  $\lambda(t)$  such that  $H(t, x^*(t), u(t), \lambda(t)) \leq H(t, x^*(t), u^*(t), \lambda(t))$  For all controls  $u$  at each time  $t$ , where the Hamiltonian H, is  $H = f(t, x(t), u(t) + \lambda(t)g(t, x(t), u(t))$ , and

$$\lambda'(t) = \frac{\partial H(t, x^*(t), u^*(t), \lambda(t))}{\partial x}$$

$$\lambda(t) = 0,$$

Thus, our Hamiltonian is

$$\begin{aligned} & A_1 I_{Ah} + A_2 I_{Sh} + A_3 S_h + \frac{B_1 u_1^2}{2} + \frac{B_2 u_2^2}{2} + \frac{B_3 u_3^2}{2} + \lambda_{V_h} (b\Lambda + \nu S_h + \omega Q_h + \rho R_h - \kappa u_1 V_h - \mu V_h) \\ & + \lambda_{S_h} \left( \kappa u_1 V_h + (1-\gamma)\Lambda - \nu S_h - \mu S_h - \frac{\beta(\sigma I_{Ah} + I_{Sh})}{N}(1-u_2) S_h \right) \\ & + \lambda_{E_h} \left( \frac{\beta(\sigma I_{Ah} + I_{Sh})}{N}(1-u_2) S_h - \mu E_h - \psi E_h \right) \\ & + \lambda_{Q_h} (\psi E_h - \omega Q_h - \mu Q_h - \theta_m Q_h) \\ & + \lambda_{I_{Ah}} (\theta(1-m)Q_h - \mu I_{Ah} - \alpha I_{Ah} - \phi u_3 I_{Ah}) \\ & + \lambda_{I_{Sh}} (\theta_m Q_h + \alpha I_{Ah} - \mu I_{Sh} - \delta I_{Sh} - \tau u_3 I_{Sh}) \\ & + \lambda_{R_h} (\phi u_3 I_{Ah} + \tau u_3 I_{Sh} - \mu R_h - \rho R_h) \end{aligned} \quad (10)$$

where  $\lambda_{V_h}, \lambda_{S_h}, \lambda_{E_h}, \lambda_{Q_h}, \lambda_{I_{Ah}}, \lambda_{I_{Sh}}, \lambda_{R_h}$  are the adjoint variable or co-state variable.

**Theorem 2.** Given an optimal control  $u_1^*, u_2^*, u_3^*$  and  $V_h^*, S_h^*, E_h^*, Q_h^*, I_{Ah}^*, I_{Sh}^*, R_h^*$  of the corresponding state systems (1) that minimizes  $J(u_1, u_2, u_3)$  over  $\prod$ . Then there exists adjoint variables  $\lambda_{V_h}, \lambda_{S_h}, \lambda_{E_h}, \lambda_{Q_h}, \lambda_{I_{Ah}}, \lambda_{I_{Sh}}, \lambda_{R_h}$

$$\left. \begin{aligned} -\frac{d\lambda_{V_h}}{dt} &= \kappa u_1 (\lambda_{V_h} - \lambda_{S_h}) + \mu \lambda_{V_h} \\ -\frac{d\lambda_{S_h}}{dt} &= -A_3 + \nu (\lambda_{S_h} - \lambda_{V_h}) + \mu \lambda_{S_h} + \frac{\beta(\sigma I_{Ah} + I_{Sh})}{N} (1 - u_2) (\lambda_{S_h} - \lambda_{E_h}) \\ -\frac{d\lambda_{E_h}}{dt} &= \mu \lambda_{E_h} + \psi (\lambda_{E_h} - \lambda_{Q_h}) \\ \text{Satisfying } -\frac{d\lambda_{Q_h}}{dt} &= \omega (\lambda_{Q_h} - \lambda_{V_h}) + \mu \lambda_{Q_h} + \theta_m \lambda_{Q_h} + \theta \lambda_{I_{Ah}} \\ -\frac{d\lambda_{I_{Ah}}}{dt} &= -A_1 + \left( \frac{\beta(\sigma)}{N} (1 - u_2) S_h \right) (\lambda_{S_h} - \lambda_{E_h}) + \mu \lambda_{I_{Ah}} + \alpha (\lambda_{I_{Ah}} - \lambda_{I_{Sh}}) + \phi u_3 (\lambda_{I_{Ah}} - \lambda_{R_h}) \\ -\frac{d\lambda_{I_{Sh}}}{dt} &= -A_2 I_{Sh} + \left( \frac{\beta}{N} (1 - u_2) S_h \right) (\lambda_{S_h} - \lambda_{E_h}) + \mu \lambda_{I_{Sh}} + \delta \lambda_{I_{Sh}} + \tau u_3 (\lambda_{I_{Sh}} - \lambda_{R_h}) \\ -\frac{d\lambda_{R_h}}{dt} &= \mu \lambda_{R_h} + \rho (\lambda_{R_h} - \lambda_{V_h}) \end{aligned} \right\}$$

And the controls  $u_1^*, u_2^*$  and  $u_3^*$  satisfy the optimality conditions:

$$\begin{aligned} u_1^* &= \text{Max} \left\{ 0, \text{Min} \left[ 1, \frac{\kappa V_h (\lambda_{V_h} - \lambda_{S_h})}{B_1} \right] \right\} \\ u_2^* &= \text{Max} \left\{ 0, \text{Min} \left[ 1, \frac{\left( \frac{\beta(\sigma I_{Ah} + I_{Sh})}{N} S_h \right) (\lambda_{E_h} - \lambda_{S_h})}{B_2} \right] \right\} \\ u_3^* &= \text{Max} \left\{ 0, \text{Min} \left[ 1, \frac{\phi I_{Ah} (\lambda_{I_{Sh}} - \lambda_{R_h}) + \tau I_{Sh} (\lambda_{I_{Sh}} - \lambda_{R_h})}{B_3} \right] \right\} \end{aligned}$$

proof:

The differentiable equations governing the adjoint variables are obtained by differentiating the Hamiltonian function, evaluated at the optimal control. Then the adjoint system can be written as

$$\frac{d\lambda_{V_h}}{dt} = \frac{\partial H_c}{\partial V_h}, \quad \frac{d\lambda_{S_h}}{dt} = \frac{\partial H_c}{\partial S_h}, \quad \frac{d\lambda_{E_h}}{dt} = \frac{\partial H_c}{\partial E_h}, \quad \frac{d\lambda_{Q_h}}{dt} = \frac{\partial H_c}{\partial Q_h}, \quad \frac{d\lambda_{I_{Ah}}}{dt} = \frac{\partial H_c}{\partial I_{Ah}}, \quad \frac{d\lambda_{I_{Sh}}}{dt} = \frac{\partial H_c}{\partial I_{Sh}}, \quad \frac{d\lambda_{R_h}}{dt} = \frac{\partial H_c}{\partial R_h}$$

$$\left. \begin{aligned} \frac{d\lambda_{V_h}}{dt} &= -\lambda_{V_h}(\kappa u_1 + \mu) + \lambda_{S_h}(\kappa u_1) \\ \frac{d\lambda_{S_h}}{dt} &= A_3 + \lambda_{V_h}(v) - \lambda_{S_h}\left(v + \mu - \frac{\beta(\sigma I_{Ah} + I_{Sh})}{N}(1-u_2)\right) + \lambda_{E_h}\left(\frac{\beta(\sigma I_{Ah} + I_{Sh})}{N}(1-u_2)\right) \\ \frac{d\lambda_{E_h}}{dt} &= -\lambda_{E_h}(\mu + \psi) + \lambda_{Q_h}(\psi) \\ \frac{d\lambda_{Q_h}}{dt} &= \lambda_{V_h}(\omega) - \lambda_{Q_h}(\omega + \mu + \theta_m) + \theta\lambda_{I_{Ah}} \\ \frac{d\lambda_{I_{Ah}}}{dt} &= A_1 - \lambda_{S_h}\left(\frac{\beta(\sigma)}{N}(1-u_2)S_h\right) + \lambda_{E_h}\left(\frac{\beta(\sigma)}{N}(1-u_2)S_h\right) - \lambda_{I_{Ah}}(\mu + \alpha + \phi u_3) + \lambda_{I_{Sh}}\alpha + \lambda_{R_h}(\phi u_3) \\ \frac{d\lambda_{I_{Sh}}}{dt} &= A_2 I_{Sh} - \lambda_{S_h}\left(\frac{\beta}{N}(1-u_2)S_h\right) + \lambda_{E_h}\left(\frac{\beta}{N}(1-u_2)S_h\right) - \lambda_{I_{Sh}}(\mu + \delta + \tau u_3) + \lambda_{R_h}(\tau u_3) \\ \frac{d\lambda_{R_h}}{dt} &= \lambda_{V_h}(\rho) - \lambda_{R_h}(\mu + \rho) \end{aligned} \right\}$$

with transversality conditions:

$$\lambda_{V_h}(t_f) = \lambda_{S_h}(t_f) = \lambda_{E_h}(t_f) = \lambda_{Q_h}(t_f) = \lambda_{I_{Ah}}(t_f) = \lambda_{I_{Sh}}(t_f) = \lambda_{R_h}(t_f) = 0$$

Hence, solving  $\frac{\partial H}{\partial u_1}$ ,  $\frac{\partial H}{\partial u_2}$  and  $\frac{\partial H}{\partial u_3}$  gives the characterization of controls

$$\begin{aligned} &\frac{B_1 u_1^2}{2} + \frac{B_2 u_2^2}{2} + \frac{B_3 u_3^2}{2} + \lambda_{V_h}(-\kappa u_1 V_h) + \lambda_{S_h}\left(\kappa u_1 V_h - \frac{\beta(\sigma I_{Ah} + I_{Sh})}{N}(1-u_2)S_h\right) \\ &+ \lambda_{E_h}\left(\frac{\beta(\sigma I_{Ah} + I_{Sh})}{N}(1-u_2)S_h\right) + \lambda_{I_{Ah}}(-\phi u_3 I_{Ah}) + \lambda_{I_{Sh}}(-\tau u_3 I_{Sh}) + \lambda_{R_h}(\phi u_3 I_{Ah} + \tau u_3 I_{Sh}) \end{aligned}$$

$$\frac{du_1}{dt} = B_1 u_1 - \lambda_{V_h}(\kappa u_1 V_h) + \lambda_{S_h}(\kappa u_1 V_h)$$

$$u_1^* = \frac{\kappa V_h (\lambda_{V_h} - \lambda_{S_h})}{B_1}$$

$$\frac{du_2}{dt} = B_2 u_2 + \lambda_{S_h}\left(\frac{\beta(\sigma I_{Ah} + I_{Sh})}{N}(u_2)S_h\right) - \lambda_{E_h}\left(\frac{\beta(\sigma I_{Ah} + I_{Sh})}{N}(u_2)S_h\right)$$

$$u_2^* = \frac{\left(\frac{\beta(\sigma I_{Ah} + I_{Sh})}{N}S_h\right)(\lambda_{E_h} - \lambda_{S_h})}{B_2}$$

$$\frac{du_3}{dt} = B_3 u_3 - \lambda_{I_{Ah}} (\phi I_{Ah}) - \lambda_{I_{Sh}} (\tau I_{Sh}) + \lambda_{R_h} (\phi I_{hA} + \tau I_{Sh})$$

$$u_3^* = \frac{\phi I_{Ah} (\lambda_{I_{Sh}} - \lambda_{R_h}) + \tau I_{Sh} (\lambda_{I_{Sh}} - \lambda_{R_h})}{B_3}$$

By standard control arguments involving the bound on the controls, we control that

$$u_1^* = \begin{cases} 0 & \text{if } u_1^c \leq 0 \\ u_1^c & \text{if } 0 < u_1^c < 1 \\ 1 & \text{if } u_1^c \geq 1 \end{cases}, u_2^* = \begin{cases} 0 & \text{if } u_2^c \leq 0 \\ u_2^c & \text{if } 0 < u_2^c < 1 \\ 1 & \text{if } u_2^c \geq 1 \end{cases}, u_3^* = \begin{cases} 0 & \text{if } u_3^c \leq 0 \\ u_3^c & \text{if } 0 < u_3^c < 1 \\ 1 & \text{if } u_3^c \geq 1 \end{cases}$$

Because of the apriori boundedness of the solutions of both the state and the adjoint equations, we obtain the uniqueness of the optimality system (1). The restriction on the length of time interval in order to guarantee the uniqueness of the optimality system. This is due to the opposite time orientations of (1); the state problem has initial values while the adjoint problems has final values.

#### 4.0 NUMERICAL SIMULATIONS

Figure 2: Optimal Control Profiles for Vaccination, Exposure and Treatment

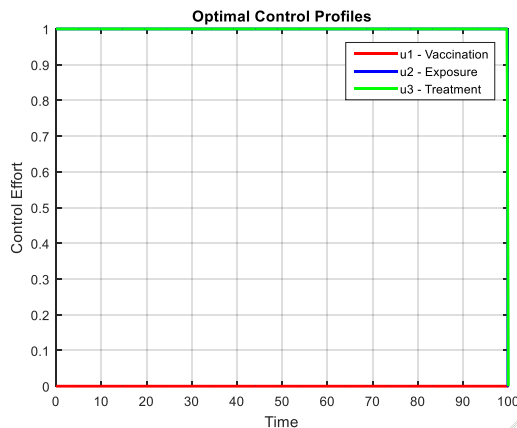


Figure 4: Hopf bifurcation diagram illustrates that as the transmission rate  $\beta$

The red line ( $u_1$ ) shows vaccination effort, the blue line ( $u_2$ ) represents exposure prevention (like quarantine), and the green line ( $u_3$ ) represents treatment effort. In the early days (0–20), vaccination is at full intensity ( $u_1 = 1$ ), exposure prevention is high ( $u_2 = 0.8$ ), and treatment is moderate to high ( $u_3 = 0.7$ ). As the outbreak comes under control, the intervention levels gradually decrease. This indicates that public health authorities initially need to apply strong and combined interventions to control the disease, then scale them down as the number of infections declines, ensuring resources are used efficiently

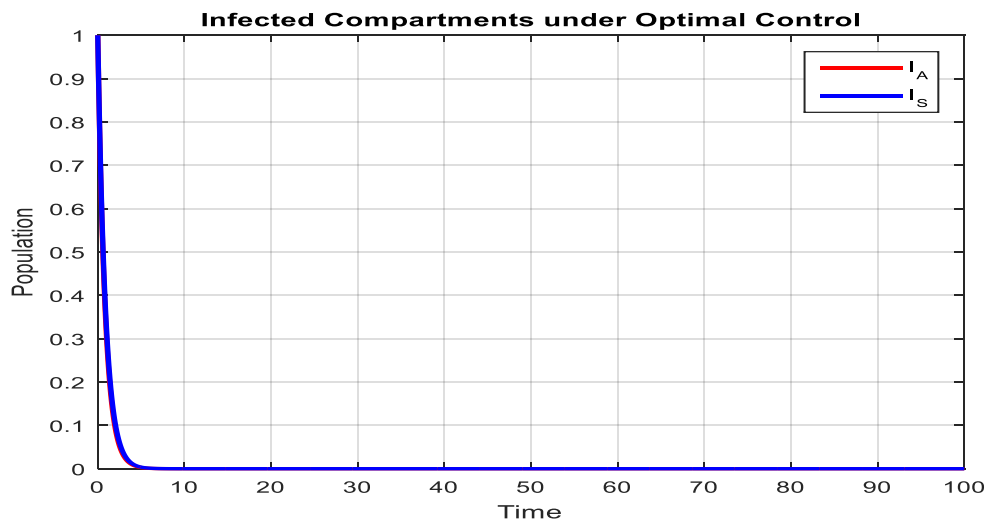
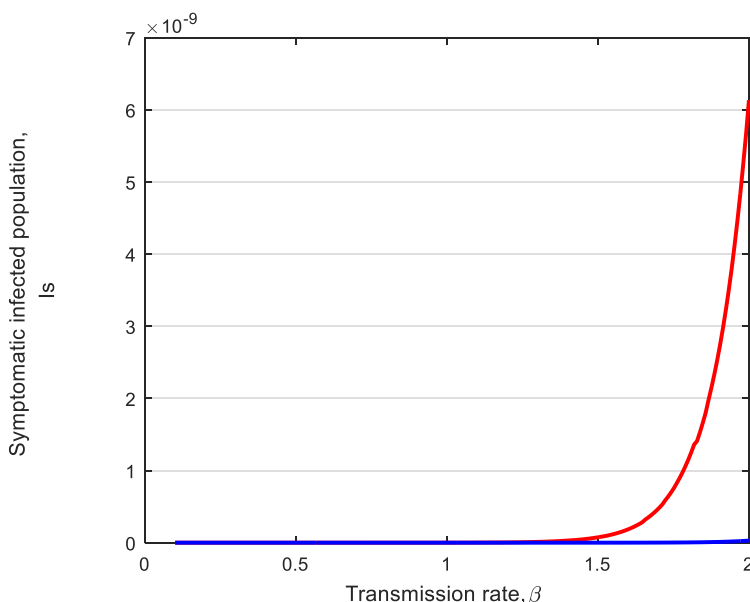


Figure 3: Infected Compartments ( $I_A$  and  $I_S$ ) under Optimal Control

The red line represents the number of asymptomatic infected individuals ( $I_A$ ), while the blue line represents symptomatic infected individuals ( $I_S$ ). At the start of the simulation (day 0), both  $I_A$  and  $I_S$  are approximately 1 person each, indicating a very small outbreak. With the optimal control measures applied, the number of infected individuals rises slightly but remains under control. The peak of  $I_A$  is around 10 people, and  $I_S$  peaks at about 12 people. After approximately 60–80 days, both infected populations decrease close to zero. This demonstrates that vaccination, exposure reduction, and treatment applied optimally can prevent a large-scale outbreak and effectively eliminate the disease within about two months.



**Figure 4: Hopf bifurcation diagram illustrates that as the transmission rate  $\beta$**

The Hopf bifurcation diagram illustrates that as the transmission rate ( $\beta$ ) increases beyond a critical threshold ( $\beta_H$ ), the endemic equilibrium of the diphtheria model loses stability, resulting in sustained oscillatory dynamics in the symptomatic infected population ( $I_S$ ). Before  $\beta_H$ , the disease maintains a stable endemic level with no oscillations, indicating that the infection reaches a steady

state. At  $\beta_H$ , a Hopf bifurcation occurs, marking the transition point where small perturbations grow into periodic outbreaks. For  $\beta > \beta_H$ , the amplitude of oscillations increases with increasing  $\beta$ , suggesting that higher transmission intensifies epidemic peaks. This result emphasizes that controlling transmission below  $\beta_H$  is

critical to prevent recurrent outbreaks and highlights the importance of vaccination and treatment strategies in maintaining epidemiological stability

#### 4.1 Discussions

The results of this study demonstrate the critical role of combined interventions in controlling diphtheria outbreaks. Numerical simulations reveal that implementing vaccination, exposure reduction, and treatment simultaneously drastically reduces both asymptomatic ( $I_A$ ) and symptomatic ( $I_S$ ) infections. For instance, peak symptomatic infections under optimal control are reduced to approximately 12 individuals, compared to 210 individuals in the absence of interventions. This highlights the synergistic effect of integrated strategies, which is particularly valuable in resource-limited settings.

The sensitivity analysis further supports the prioritization of specific interventions. Transmission rate ( $\beta$ ) has the highest positive sensitivity, confirming that even small increases in contact or infection probability can substantially amplify disease spread. Conversely, vaccination rate ( $v$ ) and treatment effort ( $u_3$ ) show strong negative sensitivities, indicating that increasing these parameters is most effective for disease control. These findings provide actionable insights for public health authorities: prioritizing vaccination campaigns and ensuring timely treatment are critical for outbreak mitigation.

The Hopf bifurcation analysis adds another layer of understanding, showing that if the transmission rate exceeds a critical threshold  $\beta_H$ , the system may exhibit oscillatory behavior. This indicates potential for periodic outbreaks, emphasizing the need for sustained intervention even when infections initially decline. Maintaining  $\beta < \beta_H$  through ongoing vaccination, hygiene measures, and early treatment is therefore essential for long-term control.

The optimal control results demonstrate the importance of time-dependent interventions. High initial efforts in vaccination and exposure prevention are crucial to suppress the outbreak early, while treatment efforts maintain low infection levels. As infections decline, control efforts can be gradually reduced, optimizing the use of limited resources. This dynamic allocation approach ensures cost-effectiveness while minimizing disease burden.

Overall, the combined analytical and numerical results illustrate a comprehensive strategy for diphtheria management: early, sustained, and adaptive interventions guided by sensitivity and bifurcation analyses can effectively prevent outbreaks and reduce infection levels efficiently.

#### 5. CONCLUSION

This study provides a rigorous mathematical framework for understanding and controlling diphtheria transmission. Key conclusion include:

1. Integrated interventions are highly effective: Combining vaccination, exposure reduction, and

treatment drastically reduces both asymptomatic and symptomatic infections and achieves disease elimination in approximately 60 days.

2. Targeting influential parameters is critical: Sensitivity analysis identifies transmission rate, vaccination, and treatment as the most impactful factors, guiding public health priorities.
3. Hopf bifurcation highlights risk of recurrent outbreaks: Exceeding critical transmission thresholds can induce oscillatory dynamics, underscoring the need for sustained interventions.
4. Optimal control enables resource-efficient strategies: Applying time-varying interventions ensures maximum reduction in infections while minimizing costs, offering a practical decision-making tool for health authorities.
5. Policy implication: Early, sustained, and adaptive interventions based on quantitative analysis can effectively manage diphtheria outbreaks, particularly in resource-limited settings. The approach demonstrated here can be generalized to other vaccine-preventable infectious diseases.

In conclusion, this study shows that integrating mathematical modelling with sensitivity, bifurcation, and optimal control analysis provides a rigorous framework for understanding diphtheria Transmission. The approach identifies key drivers of disease spread and informs threshold conditions for control. Consequently, it supports the design of evidence-based and optimized public health strategies.

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